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Skewness-Kurtosis Controlled Higher Order Equivalent Decisions

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Abstract: We define equivalence of asymptotic Gaussian expectation tests when error probabilities of first kind are approaching zero at the same restricted speed for both tests and if the same holds true for the error probabilities of second type which are measured at a moderate locally chosen alternative. To ensure such equivalence, the influence of skewness and kurtosis parameters is studied.

Keywords: Adjusted quantiles, adjusted statistics, expectation test, error probabilities, large Gaussian quantiles, moderate local alternative, process control, AMS 2010 Subject Classification 62C05, 62F03, 62E20 (Primary), 60F10 (Secondary).

1. INTRODUCTION

Asymptotic relative efficiency of one test with respect to (w.r.t.) another is extensively studied in the literature. For an introduction and overview we refer to Nikitin [1]. Several notions of efficiency may be distinguished w.r.t. how the probabilities of first and second kind test errors behave in the case of increasing sample sizes. Roughly spoken, studies of Pitman type are dealing with situations where both kinds of probabilities of test errors stabilize asymptotically at some fixed positive levels while studies of different other types take into consideration that both error probabilities are tending to zero or that one of them stabilizes asymptotically at a positive value and the other one tends to zero. Moreover, one may take into account different speeds of convergence of the two error probabilities.

The present study deals with specific situations where probabilities of both error types are tending to zero not faster than at a certain moderate speed. Note that there are different notions of large and moderate deviations in the literature of probability theory and mathematical statistics. Zons of large deviations concerned here are often called to be of the so called Linnik type, and the speed at which error probabilities are tending to zero is controlled here by a so called Osipov condition.

Second type error probabilities of a test depend on the alternatives taken under consideration. While local asymptotic normality theory is in case of sample size n concerned with differences of parameters under the hypothesis and under the alternative being of the type $C/\sqrt{n}$, here we are dealing with such differences of the type $d(n)/\sqrt{n}$ where $d(n) \to \infty$ and $d(n)/\sqrt{n} \to 0$ as $n \to \infty$. Alternatives of the latter type will be called moderate local alternatives.

Numerous statistical and probabilistic results have been derived for such situations, see Inglot and Ledwina [2, 3], Kallenberg [4, 5], Kallenberg and Ledwina [6], Ledwina, Inglot and Kallenberg [7], Richter [8 - 10] and Wood [11].

The notion of probabilities of moderate deviations partly used in those papers should be distinguished from that used for deviations in logarithmic zones, *e.g*., in Amosova [12] and Richter [8, 13, 14] for the one and multi-dimensional case, respectively.

Recent extensions of related considerations to martingale sample schemes are to be found in Fan, Grama and Liu [15].

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The paper is organized as follows. Section 2.1 provides an introduction to skewness-kurtosis adjustments of the classical asymptotic Gauss test following Richter [10], and a comparison to the well known Cornish-Fisher expansion. The equivalence of tests in the sense of Pitman is discussed in Section 2.2. The main result of this paper concerning skewness-kurtosis higher order equivalent decisions between a hypothesis and a moderate-local alternative is derived then in Section 3.

2. PRELIMINARIES

2.1. Skewness-kurtosis Adjusted Asymptotic Gauss Test

Let $X_1, ..., X_n$ be independent and identically as X distributed random variables with the common distribution law from a shift family of distributions, $P_\mu = P(\cdot - \mu)$, where the expectation equals μ, $\mu \in R$, and the variance is σ^2. Assume we are interested in deciding between a producer's hypothesis and a customer's apprehension,

$$H_0 : \mu \leq \mu_0 \text{ versus } H_{1,n} : \mu \geq \mu_{1,n} \text{ where } \mu_{1,n} > \mu_0. \tag{1}$$

The test partners are aware of the general circumstance that the values of the power function of a test are only larger than a reasonable bound if the argument of the function is chosen sufficiently far from those arguments representing the null hypothesis. They agree therefore to use a test reflecting this situation from the very beginning. The size of the gap between μ_0 and $\mu_{1,n}$ depends on what the customer may be willing to tolerate in a given practical situation, both w.r.t. the absolute value of $\mu_{1,n}$-μ and w.r.t. the costs, expressed through the sample size n.

It is well known that the statistic $T_n = \sqrt{n}\frac{\bar{X}_n - \mu}{\sigma}$ where $\bar{X}_n$ stands for the sample mean is asymptotically for $n \to \infty$ standard normally distributed, $T_n \sim AN(0, 1)$. Hence,

$$P_\mu(T_n > z_{1-\alpha}) \to \alpha, \ n \to \infty,$$

and under the $n^{-1/2}$-local non-true parameter assumption,

$$\mu_{1,n} = \mu + \frac{\sigma}{\sqrt{n}}(z_{1-\alpha} - z_\beta),$$

i.e. if one assumes that a sample is drawn with a shift of location or with an error in the variable, then

$$P_{\mu_{1,n}}(T_n \leq z_{1-\alpha}) = P_{\mu_{1,n}}(\sqrt{n}\frac{\bar{X}_n - \mu_{1,n}}{\sigma} \leq z_\beta) \to \beta \text{ as } n \to \infty,$$

where z_q denotes the quantile of order q of the standard Gaussian distribution, and α, β are from the interval (0, 1/2). Thus, the first and second type error probabilities of the decision rule $\hat{d}_0$ of the asymptotic Gauss test satisfy the asymptotic relations

$$P_{\mu_0}(\hat{d}_0 \text{ rejects } H_0) \to \alpha \ \text{ and } P_{\mu_{1,n}}(\hat{d}_0 \text{ accepts } H_0) \to \beta \text{ as } \ n \to \infty.$$

Refinements of $\hat{d}_0$ and of these two asymptotic relations where $\alpha = \alpha(n) \to 0$ and $\beta = \beta(n) \to 0$ as $n \to \infty$ were proved in Richter [10] under suitable additional assumptions.

It is often said that a random variable X satisfies the Linnik condition of order γ, $0 < \gamma < 1/2$, if

$$E_\mu \exp\{|X - \mu|^{\frac{4\gamma}{2\gamma+1}}\} < \infty. \tag{2}$$

This condition and far reaching consequences from it for probabilities of large deviations have been studied in Ibragimov and Linnik [16] (for a more general condition see Linnik [17]) and a subsequent series of papers by many authors of whom we refer here to Nagajev [18] and Richter [19] where condition (2) was fundamentally generalized in two steps.

It was proved in Richter [10] that if the conditions (2) and (3) are satisfied for a certain γ,

$$\gamma \in (\frac{s}{2s+4}, \frac{s+1}{2s+6}] \text{ where } s \in \{1,2\} \quad \textbf{(6)}$$

then the error probabilities of the adjusted decisions satisfy

$$P_{\mu_0}(\hat{d}_s \text{ rejects } H_0\ ) \sim \alpha(n) \text{ and } P_{\mu_{1,n}(s)}(\hat{d}_s \text{ accepts } H_0\ ) \sim \beta(n),\ n \to \infty. \quad \textbf{(7)}$$

These results have been equivalently reformulated in Richter [10] with the help of the first and second kind adjusted asymptotically Gaussian test statistics

$$T_n(1) = T_{n,0} - \frac{g_1}{6\sqrt{n}} T_{n,0}^2 \text{ and } T_n(2) = T_n(1) - \frac{3g_2 - 8g_1^2}{72n} T_{n,0}^3,$$

respectively. The hypothesis H_0 will be rejected according to decision rule $\hat{d}_s$ if $T_n(s) > z_{1-\alpha}(n)$, and the first and second kind error probabilities of this decision still behave as in (7).

Similar consequences for testing $H_0 : \mu > \mu_0$ or $H_0 : \mu \neq \mu_0$, as well as for constructing confidence intervals, are omitted here.

The material of this section surveys the condensed content of the basic 'testing-part' of what was presented by the author at the Conference of European Statistics Stakeholders, Rome 2014, (see Abstracts of Communication, p. 90 and Richter [10]) where, however, the equivalent 'language' of confidence estimation is used. The advanced 'testing-part' of this talk is presented in Section 3 of the present paper.

Remark 2.1. *From a formal point of view, the first and second kind adjusted asymptotic Gaussian quantiles defined in (4) and (5), respectively, have the same analytical structure as the coefficients of a Cornish-Fisher quantile-expansion (CFE), see Fisher and Cornish [23] and Bolschev and Smirnov [24]. However, CFE is valid for fixed or stabilizing values of* α *while the relations in (4) and (5) apply to the case* $\alpha(n) \to 0$ *as* $n \to \infty$*. Taking this into account, the large deviation results in their form presented here through a new light onto the CFE. Note that the CFE itself is based upon an Edgeworth type expansion of a corresponding cumulative distribution function. Theoretical and numerical comparisons of normal and large deviation approximations for tail probabilities were presented in Field and Ronchetti [25], Fu,Len and Peng [26], Ittrich, Krause and Richter [22] and Jensen [27].*

2.2. Pitman Equivalent Tests

In production process control, assume two methods are available for measuring the dimension μ of a workpiece which is serially made on a machine. In general, either methods work at different levels of costs and precision, the latter being expressed in terms of variances of measurements, σ_1^2 and σ_2^2. One may be interested then in knowing for which sample size $n_2 = n_2(n_1)$ the second method works as good as the first one works for sample size n_1. For comparing the two methods of process control being based upon the two methods of measuring a workpiece one may compare both first and second kind error probabilities when dealing with problem (1) where we are given now the i.i.d. samples $\mathcal{S}_i = \{X_1^{(i)}, ..., X_{n_i}^{(i)}\}, i = 1, 2$ satisfying $X_j^{(i)} \sim P^{(i)}(\mu, \sigma_i^2), j = 1, ..., n_i$ with $P^{(i)}(\mu, \sigma_i^2) = P_\mu^{(i)} = P^{(i)}(. - \mu)$ for any probability distribution law from a family having shift parameter equal to expectation μ, and variance σ_i^2, $i = 1, 2$. Throughout what follows in the present paper all statistics built on using the random variables $X^{(i)}_j$, as well as quantiles of their distribution functions, will be indicated by an upper subscript (i), and their higher order moments and semi-invariants as well as corresponding sample sizes will be indicated by the (possibly second) lower index i.

In this sense, $\hat{d}_0^{(i)}$ denotes the decision function based upon the asymptotically Gaussian statistic $T_{n_i}^{(i)}$ evaluated for sample $\mathcal{S}_i$ of size n_i, $i = 1, 2$.

We recall that Pitman's strategy of defining equivalence of two tests is one of several such strategies which are commonly formulated for rather general tests, see, *e.g.*, in Nikitin [1]. We restrict our consideration here, however, to pairs of tests based upon the statistics $T_{n_i}^{(i)}$ being the suitably centered and normalized means of the i.i.d.-samples

$$ {}^{(1)}_{1,n_1}(s) = {}^{(2)}_{1,n_2(n_1)}(s) \tag{13}$$

if, for $n_2(n_1) \to \infty$ *as* $n_1 \to \infty$,

$$P^{(1)}_{\mu_0}(\hat{d}^{(1)}_s \text{ rejects } H_0) \sim P^{(2)}_{\mu_0}(\hat{d}^{(2)}_s \text{ rejects } H_0) \sim \alpha(n_1) \to 0$$

and

$$P^{(1)}_{\mu^{(1)}_{1,n_1}(s)}(\hat{d}^{(1)}_s \text{ accepts } H_0) \sim P^{(2)}_{\mu^{(1)}_{1,n_1}(s)}(\hat{d}^{(2)}_s \text{ accepts } H_0) \sim \beta(n_1)$$

where $\alpha(n_1)$, $\beta(n_1)$ *satisfy the Osipov type condition (3) with* γ *chosen according to (6).*

Theorem 3.1. *Let* $X_1^{(1)}$ *and* $X_1^{(2)}$ *satisfy the Linnik condition (2) for one and the same* $\gamma = \gamma(s)$ *fulfilling (6). Moreover, let* $\alpha(n_1)$, $\beta(n_1)$ *satisfy the Osipov condition (3) for the same* $\gamma = \gamma(s)$, *and assume that*

$$\frac{n_2(n_1)}{n_1} = \frac{\sigma_2^2}{\sigma_1^2} + o(1)\frac{1}{\max\{z_{1-\alpha(n_1)}, -z_{\beta(n_1)}\}^2}, n_1 \to \infty. \tag{14}$$

If in the case $s = 1$ *skewness* $g_{1,1}$ *and* $g_{1,2}$ *of* $X^{(1)}$ and $X^{(2)}$, *respectively, satisfy*

$$\frac{g_{1,2}}{g_{1,1}} = \frac{\sigma_2}{\sigma_1}, \tag{15}$$

and if in the case $s = 2$ *additionally the corresponding kurtosis values satisfy*

$$\frac{g_{2,2}}{g_{2,1}} = \frac{\sigma_2^2}{\sigma_1^2} \tag{16}$$

then $\hat{d}^{(1)}_s$ and $\hat{d}^{(2)}_s$ are equivalent of order $s + 1$, $s \in \{1, 2\}$, respectively.

Proof. The proof of the first property of Definition 3.1 follows directly from the results in Richter [10], see Section 2.1. We recall that if condition (3) is satisfied then $x = z_{1-\alpha(n)} = o(n^\gamma)$, $n \to \infty$ for $\gamma \in (^1/_6, ^3/_{10}]$, and if (2) is satisfied then, according to Nagajev [18] and Petrov [28],

$$P_\mu(T_n > x) \sim f_{n,s}(x), x \to \infty \tag{17}$$

where

$$f_{n,s}(x) = f^{(X)}_{n,s}(x) = \frac{1}{\sqrt{2\pi}x}\exp\{-\frac{x^2}{2} + \frac{x^3}{\sqrt{n}}\sum_{k=0}^{s-1} a_k(\frac{x}{\sqrt{n}})^k\}$$

and s is an integer satisfying (6), i.e. $s = 1$ if $\gamma \in (^1/_6, ^1/_4]$ and $s = 2$ if $\gamma \in (^1/_4, ^3/_{10}]$.

Here, the constants $a_0 = \frac{g_1}{6}$, $a_1 = \frac{g_2 - 3g_1^2}{24}$ depend on skewness g_1 and kurtosis g_2 of X.

Thus, according to the construction of the skewness-kurtosis adjusted quantiles $z^{(i)}_{1-\alpha(n_1)}(s)$ and moderate local alternatives $\mu^{(i)}_{1,n_i}(s)$, $i \in \{1, 2\}$, $s \in \{1, 2\}$,

$$P^{(i)}_{\mu_{1,n_i}(s)}(T^{(i)}_{n_i} \le z^{(i)}_{1-\alpha(n_1)}(s)) \sim \beta(n_1), n_1 \to \infty. \tag{18}$$

Hence, for proving the second property of Definition 3.1 it remains to show that one can replace $\mu^{(2)}_{1,n_2(n_1)}(s)$ with $\mu^{(1)}_{1,n_1}(s)$ in (18) if there holds $i = 2$.

As a consequence of (17), and in accordance with the proof of Theorems 1 and 2 in Richter [10],

$$P^{(2)}_{\mu^{(2)}_{1,n_2(n_1)}(s)}(T^{(2)}_{n_2} \le z^{(2)}_{1-\alpha(n_1)}(s)) \sim f_{n_2,s}(z^{(2)}_{1-\alpha(n_1)}(s) - \xi_2)$$

where $f_{n_2,s} = f^{(X^{(2)}_1)}_{n_2,s}$, $\xi_2 = \sqrt{n_2}(\mu^{(2)}_{1,n_2(n_1)}(s) - \mu_0)/\sigma_2$, and

$$P^{(2)}_{\mu^{(1)}_{1,n_1}(s)}(T^{(2)}_{n_2} \le z^{(2)}_{1-\alpha(n_1)}(s)) \sim f_{n_2,s}(z^{(2)}_{1-\alpha(n_1)}(s) - \xi_1),$$

$\xi_1 = \sqrt{n_2}(\mu^{(1)}_{1,n_1}(s) - \mu_0)/\sigma_2.$

It remains therefore to show that,

$$f_{n_2,s}(z^{(2)}_{1-\alpha(n_1)}(s) - \xi_1 + [\xi_1 - \xi_2]) \sim f_{n_2,s}(z^{(2)}_{1-\alpha(n_1)}(s) - \xi_1).$$

Equivalently, we prove that,

$$f_{n_2,s}(h_s + k_s) \sim f_{n_2,s}(h_s) \quad \textbf{(19)}$$

with

$$h_s = z^{(2)}_{1-\alpha(n_1)}(s) - \sqrt{n_2}(\mu^{(1)}_{1,n_1}(s) - \mu_0)/\sigma_2$$

and

$$k_s = \xi_1 - \xi_2 = \sqrt{n_2}(\mu^{(1)}_{1,n_1}(s) - \mu^{(2)}_{1,n_2}(s))/\sigma_2.$$

According to Lemma 3.1 below, for proving (19) it is sufficient to prove that $k_s = o(h_s^{-1})$. Note that,

$$h_0 = z_{1-\alpha(n_1)} - \frac{\sqrt{n_2}}{\sigma_2}(\mu_0 + \frac{\sigma_1(z_{1-\alpha(n_1)} - z_{\beta(n_1)})}{\sqrt{n_1}} - \mu_0)$$

$$= z_{1-\alpha(n_1)}(1 - \sqrt{\frac{n_2}{n_1}}\frac{\sigma_1}{\sigma_2}) + \sqrt{\frac{n_2}{n_1}}\frac{\sigma_1}{\sigma_2} z_{\beta(n_1)}$$

and

$$k_0 = \frac{\sqrt{n_2}}{\sigma_2}(\frac{\sigma_1}{\sqrt{n_1}} - \frac{\sigma_2}{\sqrt{n_2}})(z_{1-\alpha(n_1)} - z_{\beta(n_1)})$$

$$= (\sqrt{\frac{n_2}{n_1}}\frac{\sigma_1}{\sigma_2} - 1)(z_{1-\alpha(n_1)} - z_{\beta(n_1)}).$$

By (14), $z_{1-\alpha(n_1)}(1 - \sqrt{\frac{n_2}{n_1}\frac{\sigma_1}{\sigma_2}}) = o(\frac{1}{\max})$, $\sqrt{\frac{n_2}{n_1}\frac{\sigma_1}{\sigma_2}} \to 1$ and $z_{\beta(n_1)} \to -\infty$ as $n_1 \to \infty$ where $\max = \max\,\{z_{1-\alpha(n_1)}, -z_{\beta(n_1)}\}$. It follows, symbolically, that,

$$h_0 k_0 = [O(\max)o(\frac{1}{\max^2}) + O(\max)][o(\frac{1}{\max^2})O(\max)],$$

thus $h_0 k_0 = o(1)$ as $n_1 \to \infty$. Moreover,

$$h_1 = h_0 + \frac{g_{1,2} z^2_{1-\alpha(n_1)}}{6\sqrt{n_2}} - \frac{\sqrt{n_2}\sigma_1 g_{1,1}}{\sigma_2 6 n_1}[z^2_{1-\alpha(n_1)} - z^2_{\beta(n_1)}]$$

$$= h_0 + \frac{g_{1,2} z^2_{1-\alpha(n_1)}}{6\sqrt{n_2}}(1 - \frac{\sigma_2 g_{1,1}\sigma_1^2 n_2}{\sigma_1 g_{1,2}\sigma_2^2 n_1}) + \frac{z^2_{\beta(n_1)} g_{1,1}\sigma_1\sqrt{n_2}}{6\sqrt{n_1}\sigma_2\sqrt{n_1}} g_{1,1}$$

and

$$k_1 = k_0 + \frac{\sqrt{n_2}}{\sigma_2}(\frac{\sigma_1 g_{1,1}}{6n_1} - \frac{\sigma_2 g_{1,2}}{6n_2})(z^2_{1-\alpha(n_1)} - z^2_{\beta(n_1)})$$

$$= k_0 + \frac{g_{1,1}\sqrt{n_2}\sigma_1}{6\sqrt{n_1}\sqrt{n_1}\sigma_2}(1 - \frac{\sigma_2^2 n_1 \sigma_1 g_{1,2}}{\sigma_1^2 n_2 \sigma_2 g_{1,1}})(z^2_{1-\alpha(n_1)} - z^2_{\beta(n_1)}).$$

If $s = 1$ then because of (15) and (14), $\frac{\sigma_2 g_{1,1}\sigma_1^2 n_2}{\sigma_1 g_{1,2}\sigma_2^2 n_1} = 1 + o(\frac{1}{\max^2})$ and $\frac{z_{\beta(n_1)} g_{1,1}}{6\sqrt{n_1}} = o(n_1^{-1/4})$, thus $h_1 k_1 = o(1)$. Similarly, on using (16) and (14), in the case $s = 2$, $h_2 k_2 = o(1)$.

Lemma 3.1. *If* $\varrho_{n,s}(x) = o(\frac{1}{x}), x \to \infty$ *then* $f_{n,s}(x + \varrho_{n,s}(x)) \sim f_{n,s}(x),\ x \to \infty$
Proof. Let $z = x + \varrho_{n,s}(x)$, then

$$f_{n,s}(z) = \frac{1}{\sqrt{2\pi z}} \exp\{-\frac{z^2}{2} + \frac{z^3}{\sqrt{n}}\sum_{k=0}^{s-1} a_k(\frac{z}{\sqrt{n}})^k\}$$

$$= \frac{1}{\sqrt{2\pi x(1 + \varrho_{n,s}(x)/x)}}$$

$$\exp\{-\frac{x^2(1 + \varrho_{n,s}(x)/x)^2}{2} + \frac{x^3(1 + \varrho_{n,s}(x)/x)^3}{\sqrt{n}}\sum_{k=0}^{s-1} a_k(\frac{x}{\sqrt{n}})^k(1 + \varrho_{n,s}(x)/x)^k\}$$

$\sim f_{n,s}(x)$ if (i) $x^2\frac{\varrho_{n,s}(x)}{x} \to 0$,, (ii) $\frac{x^3}{\sqrt{n}}\frac{\varrho_{n,s}(x)}{x} \to 0$ and (iii) $\frac{x^4}{n}\frac{\varrho_{n,s}(x)}{x} \to 0$. Note that assumption (i) is a stronger one than (ii) and (iii), and that it is fulfilled under the assumptions of Lemma 3.1.

CONFLICT OF INTEREST

The author confirms that this article content has no conflict of interest.

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