

Linear and Directional Domains with Cauchy Probability Distributions

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Abstract: The usual domains for Cauchy distributions have been straight lines and unit circles. These domains are closed under arbitrary changes in location and scale, whether done sequentially or simultaneously. Such closure properties have been extended to spherical Cauchy distributions. Higher dimensional Cauchy-based domains are created herein for unit hyperspheres and sets of straight lines of arbitrary dimension, and their Cauchy-like properties are determined and described. Cauchy distributions on these extended domains are shown to be closed under arbitrary transformations of location and scale, done singly or sequentially, but not generally closed when location and scale changes are done simultaneously. Stereographic projections are used to map the curved, finite surface of any hypersphere to a linear, infinite space of the same dimension as that of the hyperspherical surface. These mappings are one-one and onto, with no loss of information. These results show promise for uniting linear and directional mixtures of observations into a common domain-linear or directional.

Keywords: Euclidean space, Fisher-von Mises distribution, Generalized stereographic projection, Hyperspheres, Image analysis, Möbius mapping, Pattern recognition.

1. INTRODUCTION

Let $x_1 \dots x_n$ be a rectangular coordinate system for an *extended* Euclidean space E_n (n -space, for short). The "extended" feature means that the n -space has appended to it the single improper point ∞ . That is, $-\infty < x_j \leq \infty$ for $j = 1 \dots n$. This results, as will be seen, in the ability to transform directional data back and forth between curved and linear spaces with the same dimensionality with no loss of information, via generalized stereographic projection.

Row vectors are used for brevity: $0 = (0 \dots 0)$ is the n -vector origin in E_n and $x = (x_1 \dots x_n)$ is a typical point of E_n . The equation $(x-c)(x-c)^T = r^2$ defines a hypersphere E_n with center $c = (c_1 \dots c_n)$ and radius r .

Our focus is on unit hyperspheres, where $c = 0$ and $r = |x| = 1$. Our aim is to create and describe families of Cauchy-based distributions on linear and spherical surfaces of arbitrary dimension.

Hyperspheres are classified by the dimensionality, m , of their surfaces. A circle C has $m = 1$ but, as a courtesy, it is called a 1-sphere or S_1 . An ordinary sphere with 2-dimensional surface area is a 2-sphere or S_2 with $m = 2$, and a hypersphere with generic m -dimensional surface content is an m -sphere S or S_m .

An m -sphere needs at least $m+1$ dimensions to properly display itself and to provide the additional dimension needed for rectangular coordinates (Fig. 1). So, we put $n = m + 1$, and let n vary as m varies at will. Dimension n is not to be confused with the non-italicized north pole n of a sphere or hypersphere.

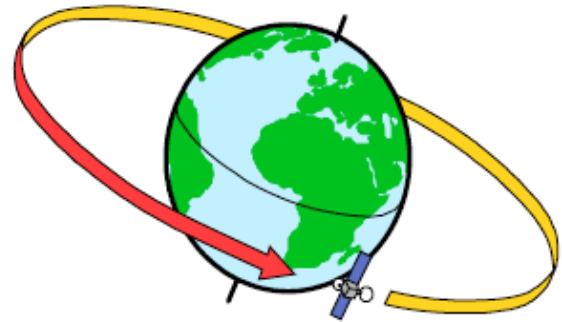


Fig. (1). Satellite orbiting a 2-sphere S_2 in E_3 . A sensor at the center of S_2 can record the spatial direction to the satellite or to any other detectable object in 3-space by the rectangular or spherical coordinates of the point where the "line of sight" from the center of the sphere to the center of the satellite intersects S_2 . Directions in n -space may be recorded as the coordinates of points on an m -sphere S_m in n -space ($n = m+1$). If the sphere were a 1-sphere S_1 in E_3 instead of a 2-sphere the universe of detectable objects would be limited to those in the plane of S_1 .

The default notation for a circle is C or S_1 , that for a d -sphere is S_d when d is not m , and that for an m -sphere is S or S_m .

Geographic terminology is used: to refer to the various hemispheres; to the equator and its equatorial m -space E_m ; to the north and south poles $n = (10 \dots 0)$ and $s = -n$; to the vertical polar axis through the poles and coincident with the x_1 -axis; to colatitude θ (the angular deviation of a point x on an m -sphere from a special fixed point on the m -sphere); to the *polar* circles passing through the poles n and s , and their horizontal centerlines L passing through the origin.

The default notation for a single centerline is L or L_1 , that for a d -space of centerlines is L_d when d is not m , while that for an m -space of centerlines is L or L_m . This m -space is a subspace of E_n , and is therefore itself an extended E_m .

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An m -sphere S generated by a sequence of rotations is created in § 2. The sequence begins by rotating a circle in 2-space into a sphere in 3-space, and ends with the generation of an m -sphere in n -space by rotation of an $(m-1)$ -sphere. Section 3 defines and describes certain terms and probability distributions, and gives some details for transforming distributions on m -sphere surfaces to and fro between rectangular $x = (x_1 \dots x_n)$ coordinates and angular or spherical coordinates $a = (r \ a_1 \dots a_m)$.

Section 4 lays the groundwork for methodology to change back and forth between linear and directional domains. Section 5 expands the methodology of § 4 to perform sequential or simultaneous transformations in location and scale for Cauchy-based probability distributions on infinite linear and finite curved objects. Section 6 briefly discusses some potential applications of the methods proposed herein.

2. CREATING M-SPHERES S_M IN E_N

Start with the unit circle S_1 in the x_1x_2 2-space of E_n , with the vertical x_1 -axis as the polar axis through $\pm n$, and the horizontal x_2 -axis as the centerline L_1 for S_1 .

Imagine being in $x_1x_2x_3$ 3-space above S_1 , looking down on the north pole n . Picture S_1 and its center line L_1 rotating CCW (counter-clockwise) through an amount π about the polar axis, and out of the x_1x_2 2-space containing S_1 into the 3-space of $x_1x_2x_3$. This rotation creates a unit 2-sphere S_2 as a 2-space surface of revolution that is a continuum of polar circles, and a continuum L_2 of their centerlines.

Continue this process by rotating S_2 and its continuum of centerlines CCW through the same amount π about the same polar axis but now out of the 3-space containing S_2 and into the 4-space of $x_1x_2x_3x_4$, thereby generating a 3-sphere S_3 and a 2-fold continuum L_3 of centerlines.

Continue this pattern, creating S_4, S_5 etc., and their associated continuums of centerlines, and finishing with the creation of S_m and an $(m-1)$ -fold continuum $L = L_m$ of centerlines.

The poles, $\pm n$, of any m -sphere S are considered as fixed and immutable, with n at the top of the polar axis, s at the bottom, and the polar axis orthogonal to the equatorial m -space E_q spanned by $x_2 \dots x_n$. The m -space L_m spanned by all the centerlines of S is the same space as the equatorial space E_q .

3. CAUCHY M-SPHERE PARAMETERS

3.1. Mean Direction μ and Rescaler δ

All Cauchy-based directional distributions dealt with herein have two parameters in addition to their dimension. These are the unit n -vector *location* parameter, μ , and the non-negative scale parameter δ , called the *rescaler*. The points x and μ on an m -sphere S have the vector forms:

$$x = (x_1 \dots x_n), \mu = (\mu_1 \dots \mu_n), \text{ with } |x| = |\mu| = 1.$$

The mean μ of a Cauchy m -sphere is the unit n -vector from the origin to the center of gravity, $E(x)$, of the probability distribution for $x \in S_m$. This center of gravity is the origin, 0, when the mean direction does not exist. We use the improper value 0 for μ to represent this, and write

$$E(x) = \rho\mu, \text{ where } 0 \leq \rho \leq 1, \text{ and } \mu \in S \Leftrightarrow \mu \neq 0.$$

The length ρ of $E(x)$ is called the *precision*. It is shown later that a Cauchy sphere S is uniformly distributed if and only if the rescaler $\delta = 1$, in which case the precision $\rho = 0$, and $\mu = 0$. By default $\delta \neq 1$ and $\rho \neq 0$. However, if $\delta = 0$ or ∞ then $\rho = 1$, and $x = \mu$ with probability one.

For circles, δ and ρ are related [1; §3.3]) by

$$\rho(\delta) = |1-\delta|/(1+\delta) = \rho(1/\delta) \text{ for any } \delta > 0.$$

This formula changes with m , becoming more complicated for larger m , e.g.: when $m = 2$, $\rho = \coth \kappa - 1/\kappa$ where $\delta = \exp(\kappa)$ [1; §4.4].

3.2. Quick Definition of S_m Distribution

The m -sphere S is said to be *Cauchy-based* when the variable part of its probability density function (also called the profile graph function, or pgf for short) has the form: $pgf(\theta) \propto (A - B \cos \theta)^{-1}$, in which case we write:

$$S, \text{ or } x, \text{ or } \theta \sim C_m(\mu, \delta) \tag{1}$$

where A and B are constants and functions of δ ; $\cos \theta = x\mu^T$ where x is a point on S , μ is the mean direction for x , and θ is the colatitude for x . In such case we say S , or x , or θ is Cauchy μ, δ . When S has a uniform distribution we say S is Cauchy 0, 1, and write: S , or x , or $\theta \sim C_m(0,1)$.

3.3. Transforming Coordinate Systems

The equations given in this section for transforming spherical to rectangular coordinates for points on m -spheres were adapted from [2]. They are used below to derive the Jacobian matrix J for the transformation.

Assume for the moment that the radius r of S is arbitrary. For brevity put $s_d = \sin a_d$ and $c_d = \cos a_d$, for $d = 1-m$. The elements of x are given, in terms of those for the spherical coordinates in $a = (r \ a_1 \dots a_m)$, by:

$$x_1 = r \times c_1; \ x_d = r \times s_1s_2 \dots s_{d-1}c_d, \text{ for } d = 2-m; \ x_n = r \times s_1s_2 \dots s_{m-1}s_m.$$

Equations for expressing the spherical coordinates of a in terms of the rectangular coordinates of x are given in [2]. The angle a_m given by $a_m = 2 \cot q$, where $q = \{(x_m^2 + x_n^2)^{1/2} + x_m\}/x_n$, plays the role of a longitude for S . For further details see [2].

3.4. Jacobian of the Transformation

The radius r of S is allowed to vary. S has a uniform distribution on its surface and rectangular coordinates x . The joint *probability density function* (*pdf*, for short) for x , when $S \sim C_m(0,1)$, is: $pdf(x) = a$ a norming constant. The joint *pdf* (x) is to be transformed to the more tractable joint *pdf* (a) of the angular coordinates in a for S . The $n \times n$ Jacobian matrix J of partial derivatives is defined by its row-column element k,d , or its column d , from

$$J = (\partial x_k / \partial a_d), \text{ or } J = (\partial x / \partial a_0 \ \partial x / \partial a_1 \ \dots \ \partial x / \partial a_m) \tag{2}$$

The change of variables theorem [3] provides that the joint *pdf* (a) is related to the joint *pdf* (x) for all x , except the origin 0, by:

$$pdf(a) = pdf\{x(a)\} |\det J| \tag{3}$$

where $\det J$ is found to be

$$\det J = a_0^m \sin^{m-1} a_1 \sin^{m-2} a_2 \dots \sin^1 a_{m-1} \quad (4)$$

Now fix $a_0 = r = 1$. Since the distribution on S is uniform, $pdf\{x(a)\}$ is also just a norming constant, say N , and (3) can be expressed as

$$pdf(a) = \Pi_d pdf(a_d) = \Pi_d (N_d \sin^{m-d} a_d), d = 1-m \quad (5)$$

with N_d a norming constant for a_d and $N = N_1 \times \dots \times N_m$. The m^{th} sine term of the second product in (5) has disappeared since $\sin^{m-m} a_m \equiv 1$. Conclude from (5) that the m angles a_j are jointly independent and uniformly distributed over their respective surfaces, which fixes the norming constants N_d and N .

3.5. Cauchy m -Sphere Distributions

An m -sphere S with a Cauchy distribution on its surface will be called a *Cauchy m -sphere*. It is defined by its probability density element (*pde*, for short), in (8) below. Let S be a Cauchy m -sphere with mean μ and rescaler δ , and put, as in § 3.2:

$\cos \theta = x\mu^T, 0 \leq \theta \leq \pi$; and $x, \mu \in S$, where θ is the (6) colatitude.

We say S or θ or x is *Cauchy* μ, δ , and write:

$$S \sim C_m(\mu, \delta), \text{ or } \theta \sim C_m(\mu, \delta), \text{ or } x \sim C_m(\mu, \delta) \quad (7)$$

when the *pde* for the colatitude θ has the form:

$$pde(\theta; \mu, \delta) = K_m(A - B \cos \theta)^{-1} \sin^{m-1} \theta d\theta \quad (8)$$

where A and B are functions of δ (9), and K_m is a norming constant which is a function of m and δ :

$$A = (\delta^2 + 1)/(2\delta) > B = (\delta^2 - 1)/(2\delta) \geq 0 \quad (9)$$

Equations (6)-(8) show that the $C_m(\mu, \delta)$ distribution is completely described by that for the scalar colatitude θ . Formulae for integrating (8) to get K_m can be found in [4].

The *probability density function* (*pdf*, for short) for θ is just *pde*(θ) without the $d\theta$:

$$pdf(\theta; \mu, \delta) = K_m(A - B \cos \theta)^{-1} \sin^{m-1} \theta, \quad (10)$$

and a *profile graph* function (*pgf*) for θ is just *pdf*(θ) without the $\sin^{m-1} \theta$:

$$pgf(\theta; \mu, \delta) \propto (A - B \cos \theta)^{-1} \quad (11)$$

By (9) and (11) the $C_m(\mu, \delta)$ distribution degenerates to $C_m(0, 1)$ if and only if the rescaler δ is unity.

The *pgf* has the same form for all d -spheres, regardless of their surface dimensions or mean directions. A similar constancy of the *pgf* across different dimensions holds for the totality of Fisher-von Mises distributions [5], wherein *pgf*(θ) = $\exp(\kappa \cos \theta)$.

Any function proportional to (11) also serves as a *pgf* for θ . Knowing m and the form of one of the *pde*, *pdf* or *pgf* functions of (8), (10) or (11), respectively, is equivalent to knowing all three forms.

Contours of equal *pgf* values are the parallel ($m-1$)-spheres of latitude centered about the mean axis. These con-

tour values decrease monotonically from a maximum at the mode μ , where $\theta = 0$, to a midpoint when $\theta = \pi/2$, to a minimum at $-\mu$, where $\theta = \pi$. On the other hand the *pdf* values when $m \geq 2$ are zero at $\theta = 0$, then climb rapidly, then decrease slowly to 0 again at $\theta = \pi$. Thus the *pgf* gives a more realistic impression of the actual probability distribution. See [6] for a graphic example.

Each $C_m(\mu, \delta)$ distribution is isotropic about its mean axis, $\pm \mu$. Watson [7] has developed an array of statistical methods for analyzing data samples from isotropic directional distributions of arbitrary dimensions that is suitable for $C_m(\mu, \delta)$ distributions.

Cauchy m -sphere distributions possess an interesting duality feature, namely:

$$\theta, S, x \infty C_m(\mu, \delta) \Leftrightarrow \theta, S, x \infty C_m(-\mu, \delta^{-1}) \quad (12)$$

To see this, specify the argument of A as a function of δ or δ^{-1} by $A(\delta)$ or $A(\delta^{-1})$ and likewise for B , in (9). Then it is readily verified that $A(\delta) - B(\delta) x\mu^T = A(\delta^{-1}) - B(\delta^{-1}) x(-\mu)^T$, and (12) follows.

Another interesting feature is that $A+B = \delta$ and $A-B = \delta^{-1}$ which, in conjunction with (11), can be used to find the ratio of the maximum *pgf* values to their corresponding minimum values for any Cauchy $C_m(\mu, \delta)$ distribution (the ratio is either δ^2 or δ^{-2} according as $\delta > 1$ or $\delta < 1$). This ratio is the same (δ^2 or δ^{-2}) for all dimensions m and any mean direction μ on S_d . The corresponding ratio for the family of Fisher-von Mises distributions is $\exp(2\kappa)$ for all dimensions and all mean directions (see [8] for relevant properties of the Fisher-von Mises distributions).

4. STEREOGRAPHIC PROJECTION

4.1. Rotating m -Spheres into Polar Positions

At times it is necessary to rotate μ to a polar mean position $\pm n$, which is the fixed position for an m -sphere S with the added proviso that $\mu = \pm n$, so that

$$S \sim C_m(\pm n, \delta), \text{ say.}$$

The transpose u^T of any input point u (except $u = \pm n$) on S is rotated to the transpose v^T of the output point v of any other given point v on S by the $n \times n$ orthogonal matrix Q given by

$$Qu^T = \{(u+v)^T(1+uv^T)^{-1}(u+v) - I_n\}u^T = v^T \quad (13)$$

where I_n is the $n \times n$ identity matrix. The orthogonal matrix Q is symmetric and thus equal to its own inverse. It may, among other things, be used to rotate μ into or out of polar mean positions.

The poles, $\pm n$, of any m -sphere S are considered as fixed and immutable, with n at the top of the polar axis, s at the bottom, and the polar axis orthogonal to the equatorial m -space E_q spanned by $x_2 \dots x_n$. It is necessary, before doing any projections, to ensure this positioning of the polar axis and the equatorial m -space E_q . It may also be necessary to rotate μ to n .

4.2. Projective Mappings, Images, and Spaces

The equator of S is the unit ($m-1$)-sphere orthogonal to the polar axis of S . The equatorial m -space E_q of S is the set

of points y with the form $(0\ x_2\dots x_n)$ that are orthogonal to the polar x_1 -axis, the exact same set of points as the m -space L (or L_m) spanned by the centerlines of the polar circles of S . All these linear point sets are extended sets, with the single improper point ∞ appended to each and every line in them. All parallel lines of L are extended lines, and they all meet at infinity; indeed, *all* the lines of L are extended and meet at infinity. For more details about the improper point ∞ see [9-12].

In this section stereographic projection is extended beyond the usual 2- and 3-dimensional mappings between circles and lines, and spheres and planes. We use real n -vectors, x and y , on m -spheres S and on their centerline m -spaces L . We will construct a projective mapping P_{LS} from S to L : $L = P_{LS} S$, which projects the unit vectors x on S to the points y of L in a one-one and onto fashion: $y = P_{LS} x$. Since this projective mapping must be one-one and onto, it implies the consequent result that $L = P_{LS} S$, with inverse mapping given by $S = P_{SL} L$, say. In view of the above we may say that L and S (and x and y) are projective *images* of one another, and write:

$$[L \leftrightarrow S \Leftrightarrow L = P_{LS} S] \Leftrightarrow (y = P_{LS} x \text{ and } x = P_{SL} y) \quad (14)$$

in a one-one and onto fashion, for all x in S and y in L . A more precise definition of "image" is this:

For the projective mapping $y = P_{LS} x$ and its implied result $L = P_{LS} S$, the points x and y are called projective images (or images, for short) of one another, and we write: $y \leftrightarrow x$, when x and y are collinear with the north pole n . If (n,x,y) are collinear then so are (n,y,x) , so we may also write $x \leftrightarrow y$.

The fixed north pole n on S is called the *projection point* of the mapping, And the image relation is an equivalence relation, being reflexive, symmetric and transitive.

All points on S or L or E_n are row n -vectors. The m -sphere S has definition $S = \{x: x = (x_1\dots x_n), |x| = 1\}$ (ext), and $S = \{a: a = (1, a_1\dots a_m)\}$ (int, because it is expressed in terms of its internal spherical coordinates a).

L has extrinsic definition: $L = \{y: y = (0x_2\dots x_n)\}$ (ext) and intrinsic definition $L = \{y: y = (0y_2\dots y_n)\}$ (int). The extended linear n -space is defined by $E_n = \{x: x = (x_1\dots x_n), -\infty < x_j \leq \infty\}$ (int).

We proceed to derive the projective mapping P_{LS} which maps every point x on the finite curved m -sphere S , in a one-one and onto manner, to a point y on the infinite linear m -space of L .

4.3. A Projective Role for Cauchy Distributions

Any unit circle C can be stereographically projected to its extended centerline L by the one-one and onto mapping $y = \cot a/2$, where $y \in L$, $x = (\sin a, \cos a) \in C$, and a is a suitably defined angular deviation of x from n . The value of $y \in (-\infty, \infty]$ as x moves around the circle. This example is extended to bring linear and circular Cauchy distributions into the picture. Recall that a point y varying over a line L , and/or L itself, are said to be linear Cauchy distributed, with *median* v and scale parameter the rescaler δ , over the centerline L , and we write: y or $L \sim L_1(v, \delta)$, when y has probability density function

$$\text{pdf}(y) = (\delta\pi)^{-1} [1 + \{(y-v)/\delta\}^2]^{-1} \text{ where } \delta > 0 \quad (15)$$

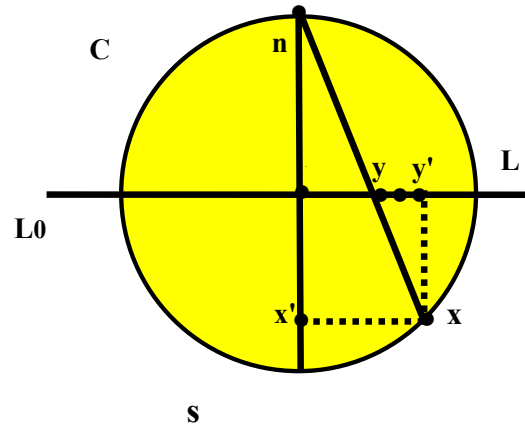


Fig. (2). Planar Cross-Section of the m -sphere S and centerline m -space L for the Projective Mapping $y = P_{LS} x$. The cross-section of the m -sphere S is the unit circle C , and the cross section of the m -space L is the horizontal centerline L . The three points n, x, y in Fig. 2 are collinear, so $x = (x_1\dots x_n)$ (ext) and $y = (0y_2\dots y_n)$ (int) are images, and we write: $x \leftrightarrow y$.

We say y or L is *standard Cauchy* and write: y or $L \sim C_1(0,1)$, when the median v is zero and the rescaler is unity. It is well-known that

$$L \leftrightarrow C \Rightarrow \{L \sim L_1(0,1) \Leftrightarrow C \sim C_1(0,1)\} \quad (16)$$

This states that if a projective mapping P_{LC} exists between a unit circle C and its extended centerline L , then a uniform distribution on C induces a standard Cauchy distribution on L , and conversely. Downs [1,§3.1] used the change of variables theorem to generalize (16) to

$$L \leftrightarrow C \Rightarrow [L \sim L_1(0,\delta) \Leftrightarrow C \sim C_1(\mu_\delta, \delta)] \quad (17)$$

where $\mu_\delta = (s,0,n)$ according as δ is ($<1,=1,>1$).

This is an interesting result. It states that, if L and C are images of one another as in (16), then rescaling L by δ induces a Cauchy μ_δ, δ distribution on C , and vice versa, thus creating a non-uniform Cauchy distribution from scratch with (17) via an unadulterated line and circle in (16). We can even use (17) to create an arbitrary circular Cauchy μ, δ from scratch by rotating μ_δ in (17) to μ , via (13).

To extend these results, define the m -space L to be Cauchy v, δ over an extended m -space L , and write:

$$L \sim L_m(v, \delta) \text{ when every line } L \text{ in } L \text{ is } L_1(v, \delta).$$

We claim now– and prove later–a remarkable extension of (17):

$$L \leftrightarrow S \Rightarrow [L \sim L_m(0,\delta) \Leftrightarrow S \sim C_m(\mu_\delta,\delta)] \quad (18)$$

This result enables us to convert an infinite linear m -space L to a finite curved m -space S , and conversely, for any positive integer m . However, we must first prove the antecedent that S and L are projective images: $S \leftrightarrow L$. We proceed to do just that.

We seek a projective mapping P_{LS} which sends every point x of an m -sphere S to a point y in an m -space L in a one-one and onto manner, so (14) is satisfied and L and S are indeed projective images of one another. First, simplify the complicated m -spaces of L and S to 1-spaces L and C using

the fact that all the action stems from the three special points n, s, x on the surfaces of S and the unit circle C (Fig. 2).

The three special points are the poles n and s of S and C , and a randomly selected non-polar point x on S . These special points fix the plane containing C , and serve to define C as the unique polar circle cross-section of S which contains x . The centerline L of C is the corresponding cross-section of L . The point x on S has unit length and so must be on C . To construct C draw a vertical polar axis for S and for C two units in length. Label the top, middle and bottom points as $n, 0$ and s (north pole, origin, and south pole), thus determining C (Fig. 2). The m -sphere S has been reduced to the cross-section of S containing the four points n, s, x and y . The equatorial m -space $E_q = L_m$ has been reduced to the centerline L of C .

The perpendicular projections of x onto the polar axis $\pm n$ and the line L yield the auxiliary points x' and y' respectively. From the rectangle with vertices $0, x', x, y'$ we get $x = x' + y'$, or $(x_1 \dots x_n)$ (ext) = $(x_1 0 \dots 0)$ (ext) + $(0 x_2 \dots x_n)$ (ext). Note that $x_1 < 0$ since x is below L , and that $y - y' \parallel x - x'$. As x varies over S , the circle C and its image line L change correspondingly. The points y of each centerline are mapped one-one and onto the points x of its corresponding polar circle, thanks to the extension to ∞ of the lines of L .

If x lies above L then so does x' , and x is then between n and y , with y still on L but outside of C . But the fundamental structure of the point configuration of Fig. (2) remains intact. Analogous configurations occur when x is to the left of the polar axis.

Some features of the mapping P_{LS} become apparent. The north pole n is mapped to the improper point ∞ of L , and the south pole s to the origin 0 . The equator of S is mapped to itself. Since E_q is orthogonal to the polar axis of S_m then every line L through 0 in E_{qm} must be orthogonal to the polar axis.

The mapping $y = P_{LS} x$ sends the point x on the finite m -dimensional curved surface of S to the point y on the infinite m -dimensional linear space L , and conversely for P_{SL} . The point $x = (x_1 \dots x_n)$ on S_m is restricted by $xx^T = 1$, but its image point $y = (0 y_2 \dots y_n)$ on E_q is unrestricted. The first coordinate y_1 of y is always 0 for all x .

We seek an extrinsic form for y as a function of x . To this end we derive the mappings P_{SL} and P_{LS} to prove their existence. For P_{LS} this is done by expressing each element of y as a function of the elements of x , and conversely for P_{SL} . We show, with heavy reliance on Fig. (2), that:

$$y = (0 y_1 \dots y_n) (int) = P_{LS} x = y' / (1 - x_1) (ext) \tag{19}$$

where $y' = (0 x_2 \dots x_n)$. Note the equation $x_1 = 0$ is an extrinsic definition of the equatorial m -space $L = E_q$.

To show (19) observe that the right triangles with hypotenuses through the points (n, y) and (n, x) are similar, so $|y|/|n| = |x - x'| / (|n| + |x'|)$, which is the same as $|y|/1 = |y'| / (1 - x_1)$ (because x_1 is negative), and this is the same as $|y|/|y'| = 1 / (1 - x_1)$. The points x, y and y' are always on the same side of the polar axis, implying that y and y' are always parallel with the same sense (both east of 0 in Fig. 2), implying equality of the unit n -vectors: $y/|y|$ and $y'/|y'|$. Since

$|y|/|y'| = 1 / (1 - x_1)$, we infer that $y = y' \times |y|/|y'| = y' / (1 - x_1)$, which proves (19). Observe that (19) is valid regardless of the quadrant in which x lies on C .

The final step is to find, for each image pair (x, y) , an extrinsic representation of x in terms of intrinsic elements of L for the inverse mapping P_{SL} . Let the point $y = (0 y_2 \dots y_n) \in L_m$ be assigned. We must find

$$x (ext) = P_{SL} y, \text{ where } y = (0 y_2 \dots y_n) (int).$$

Assume the reasonable conjecture that the form for $x = P_{SL} y$ when $n = 3$ is the same as that when n is arbitrary. If n is 3 the known form (see [9]) is:

$$x = P_{SL} y = (q^2 - 1 \ 2y_2 \ 2y_3) / (q^2 + 1), \text{ and } q^2 = y_2^2 + y_3^2.$$

For arbitrary n , the form is:

$$x = P_{SL} y = (q^2 - 1 \ 2y_2 \dots 2y_n) / (q^2 + 1), \quad q^2 = \sum y_j^2 \tag{20}$$

where the sum is for $j = 2-n$. Initially test the conjecture by checking that $|x|^2 = 1$. By (20)

$$xx^T = \{(q^2 - 1)^2 + 4y_2^2 + \dots + 4y_n^2\} / (q^2 + 1)^2.$$

The numerator of this is $(q^2 - 1)^2 + 4q^2 = q^4 - 2q^2 + 1 + 4q^2$ which is $(q^2 + 1)^2$, the denominator. The quick initial test is met. The decisive test is: does $P_{SL} P_{LS} x = x$? Since $x' = (0 \ x_2 \dots x_n)$, then

$$y = P_{yx} x = y' / (1 - x_1) (ext) = (0 x_2 \dots x_n) / (1 - x_1) (ext) \text{ by (19). By the conjecture (20) we have:}$$

$$x = P_{SL} y = (x_1 \dots x_n) = (q^2 - 1 \ 2y_2 \dots 2y_n) / (q^2 + 1) \tag{21}$$

with $q^2 = y_2^2 + \dots + y_n^2$, implying

$$x_1 = (q^2 - 1) / (q^2 + 1), \text{ and } y_j = x_j (1 - x_1) \text{ for } j = 1-n.$$

Solve the equation $x_1 = (q^2 - 1) / (q^2 + 1)$ for q^2 and get

$$q^2 = (1 + x_1) / (1 - x_1), \text{ so } q^2 + 1 = 2 / (1 - x_1), \text{ and, by (21)}$$

$$2y_j / (q^2 + 1) = \{2x_j / (1 - x_1)\} \times \{(1 - x_1) / 2\} = x_j, \tag{22}$$

for $j = 2-n$, confirming that $x = P_{SL} P_{LS} x$ and completing the proof that the m -spaces S and L are projective images of one another. This satisfies the antecedent condition in (18) that $S \leftrightarrow L$, and sets the stage for proving (18).

4.4. Proof of $L \leftrightarrow S \Rightarrow [L \sim L_m(0, \delta) \Leftrightarrow S \sim C_m(\mu_\delta, \delta)]$

We must prove $L \sim L_m(0, \delta) \Leftrightarrow S \sim C_m(\mu_\delta, \delta)$, given that $L \leftrightarrow S$. First assume that $L \leftrightarrow S$ and $S \sim C_m(\mu_\delta, \delta)$. We must prove that $L \sim L_m(0, \delta)$:

Proof. The set of centerlines L in L span L ; the set of polar circles C in S span S ; every L in L has a unique polar circle C in S : the lines and circles are paired off as (L, C) ; and the continuum of (L, C) pairs span the pair of m -spaces (L, S) .

Since $S \sim C_m(\mu_\delta, \delta)$ for all pairs (L, C) the C s must be identically distributed as $C_1(\mu_\delta, \delta)$ with identical mean $(s, 0, \text{ or } n)$ by (11). Then the corresponding L s for the C s are identically distributed as $L_1(0, \delta)$ by (17), and L is $L_m(0, \delta)$ by definition.

Assume $L \sim L_m(0, \delta)$. We now show $S \sim C_m(\mu_\delta, \delta)$:

Proof. The centerlines L of L are identically distributed as $L_1(0, \delta)$ by definition of $L_m(0, \delta)$. Then, by (17), $C \sim C_1(\mu_\delta, \delta)$ for every polar circle C on S . This implies the m -sphere S

has profile generating function $pgf(S) \propto (A-Bx\mu_\delta^T)$ where $x \in S$, implying by (11) and the discussion following it that $S \sim C_m(\mu_\delta, \delta)$. This completes the proof of (18).

There are two fixed points for S_m and two for L_m in every stereo-projective mapping between them: the fixed points for S_m are the poles n and s on S_m , and their respective images are the fixed points ∞ and 0 of L_m , since these fixed points are inviolate and cannot be changed.

The crucial role of ∞ in this process legitimizes and solidifies the conclusions that the projective mapping from the curved surface of S_m to the linear surface of L_m is indeed a one-one and onto mapping.

5. TRANSFORMATIONS

Changing μ to μ' , say, is done by rotating μ^T to μ'^T . Changes in scale from δ to δ' are done by rescaling the coordinate axes from δ to 1, then from 1 to δ' , or by simply rescaling the $x_2 \dots x_n$ coordinate axes by (δ'/δ) . A variety of transformations can be achieved by deft manipulation of (16)-(18).

Circular Cauchy distributions are amenable to sequences of simultaneous transformations of location and scale parameters because a special class of complex-valued Möbius transformations form a subgroup under composition of transformations. This subgroup is composed of Cauchy circle mappings from one planar unit circle to itself or to another unit circle, with the resulting circular Cauchy distributions having transformed parameters. For further details using complex variables see [1; § 2, 3, 4].

The spherical Cauchy distributions are closed under all Möbius transformations, with results analogous to those for circular Cauchy distributions. For further details (in a complex plane setting) see [1; § 2, 4, 4.8].

The evidence is mounting that *a family of spherical distributions is closed under Möbius transformations if and only if that family is Cauchy-based.*

Analogous evidence has accumulated for circular Cauchy distributions when the Möbius mappings are confined to the subgroup of circle mappings. See [1, § 2 & 4] for more details.

6. DISCUSSION

The techniques of pattern recognition and image analysis are heavily dependent on sophisticated statistical methods, large databases and extensive computations. Bahlmann [13] has pointed out situations where directional and linear data concurrently appear in these fields but where there is no integrated body of statistical techniques that provide an integrated solution to this problem of disparate data domains. The family of Fisher-von Mises distributions has been extended from domains of observations of vectors on m -

spheres, to domains where the observations are rigid configurations of p distinguishable directions in n -space, $p \leq n$ [14]. He applied this successfully to analyses of the orientations of spatial QRS loops observed from clinical vectorcardiograms, each with 2 vectors in 3-space ($p = 2, n = 3$). These methods show promise for application in such areas as pattern recognition and image analysis that possess endless varieties of shapes and configurations.

The Cauchy families abound in hyperspherical and linear domains, and Cauchy objects in either of these domains can be simply transformed to Cauchy objects in the other domain, thus eliminating the problem of disparate data domains. All Cauchy-based objects in either domain are amenable to changes in location and scalar parameters.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflicts of interest.

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