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A Bivariate Law of Iterated Logarithm for Partial Sums and Delayed Sums

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Abstract: When the random variables are positive strictly stable, we obtain a bivariate law of iterated logarithm for the vector of partial sums and delayed sums.

Key Words: Law of iterated logarithm, bivariate summands, stable laws.

1. INTRODUCTION

Let $\{X_n, n\geq 1\}$ be a sequence of independent identically distributed (i.i.d) positive strictly stable random variables (r.v.s) with exponent α, $0 < \alpha < 1$. Set $S_n = \sum_{k=1}^{n} X_k$ and $T_n = S_{n+a_n} - S_n$, where $\{a_n\}$ is non – decreasing sequence of positive integers. Write $\xi_n = \left\{ \left(\frac{S_n}{n^{1/\alpha}}\right)^{\theta_n}, \left(\frac{T_n}{a_n^{1/\alpha}}\right)^{\gamma_n} \right\}$, where

$\theta_n = (\log\log n)^{-1}$ and $\gamma_n = \left(\log \frac{n}{a_n} + \log\log n\right)^{-1}$.

When $(S_{1,n})$ and $(S_{2,n})$ are independent copies of (S_n), the authors in [1] have obtained the set of all limit points of the sequence $\left\{ \left(\frac{S_{1,n}}{n^{1/\alpha}}\right)^{\theta_n}, \left(\frac{S_{2,n}}{n^{1/\alpha}}\right)^{\theta_n} \right\}$. In this paper, under different conditions on (a_n),

we obtain the almost sure limit sets of the sequence $\{\xi_n, n \geq 1\}$. A careful observation tells that the limit sets change with the rate of growth of a_n in comparison with n.

The LIL in this paper is based on the right tail of the d.f. or the probability of occurrences of large values (following power law) in spirit, is on the lines of [2]. In [3], LIL has been obtained for $\left\{ \frac{S_n}{\alpha_n n^{1/\alpha}}, \frac{T_n}{\beta_n a_n^{1/\alpha}} \right\}$, for suitable choices of α_n and β_n, which depend on the behavior of the d.f. near the tail approaching zero (exponentially fast). As such, the normalization in [3] is linear and in the present paper, it is power normalization.

Through out this paper [x] stands for the largest integer which is less than or equal to a positive number x, where as a.s and i.o mean almost surely and infinitely often respectively. C, ε (small), k (integer) and N (integer), with or with out a suffix, stand for positive constants. For any sequence (Y_n) of r.v.s, lim sup (inf) $Y_n = \alpha$ (β) is to be read as lim sup $Y_n = \alpha$ and lim inf $Y_n = \beta$.

In the next section we present some preliminary results. The almost sure limit sets of the vector sequence $\{\xi_n, n \geq 1\}$ are obtained in the last section. We assume that $\frac{a_n}{n} \sim b_n$, where (b_n) is non-increasing. For instance, if $a_n = [n^p]$, $0 < p < 1$ then $\frac{a_n}{n} = \frac{[n^p]}{n}$. Taking $b_n = \frac{n^p}{n}$, one can see that $\frac{a_n}{n} \sim b_n$ and (b_n) is non-increasing. However, taking p = ½, one can observe that $\frac{a_n}{n}$ fails to be non-increasing. Similar justification holds when $a_n = [np]$. Here $b_n = \frac{np}{n} = p$, $0 < p < 1$.

2. LEMMAS

Lemma 1 (Extended Borel – Cantelli Lemma)

Let (E_n) be a sequence of events in a common probability space. if (i) $\sum_{n=1}^{\infty} P(E_n) = \infty$ and

(ii) $$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{s=1}^{n} P(E_k \cap E_s)}{\left(\sum_{k=1}^{n} P(E_k)\right)^2} \geq C,$$ then $P(E_k \text{ i.o}) \geq C^{-1}$.

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For proof, see [4, lemma P3, p.317].

Lemma 2

Let (A_n) be a sequence of events in a common probability space. If $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P\left(A_n \cap A_{n+1}^c\right) < \infty$ then $P(A_n \text{ i.o}) = 0$. For proof see [5, lemma 1*, p.385].

Lemma 3

$$\limsup_{n\to\infty} (\inf) \left(\frac{S_n}{n^{1/\alpha}}\right)^{\theta_n} = e^{1/\alpha}(1) \quad \text{a.s}$$

For proof see [1]

Lemma 4

Let $\{X_n, n\geq 1\}$ be i.i.d positive strictly stable r. v. s with exponent α, $0 < \alpha < 1$. Let (a_n), $0<a_n\leq n$, be a sequence of non - decreasing integers with $\frac{a_n}{n} \sim b_n$, where b_n is non - increasing. Then

$$\operatorname*{Lim}_{n\to\infty} \operatorname{Inf} \left(\frac{T_n}{a_n^{1/\acute{a}}}\right)^{\tilde{a}_n} = 1 \qquad \text{a.s}$$

Proof

To prove the lemma it suffices to show that for any $\varepsilon > 0$,

$$P\left(\frac{T_n}{a_n^{1/\alpha}} \leq \left(\frac{n}{a_n} \log n\right)^{\varepsilon} \text{ i.o}\right) = 1 \tag{1}$$

and

$$P\left(\frac{T_n}{a_n^{1/\alpha}} \leq \left(\frac{n}{a_n} \log n\right)^{-\varepsilon} \text{ i.o}\right) = 0 \tag{2}$$

The fact that X_n's are positive valued strictly stable r.v.s implies that $\frac{T_n}{a_n^{1/\alpha}}$ and X_1 are identically distributed. Observe that $\frac{n}{a_n}$ is non-decreasing implies $\left(\frac{n}{a_n} \log n\right)^{\varepsilon} \to \infty$ as $n \to \infty$. We therefore have $P\left(T_n \leq a_n^{1/\alpha} \left(\frac{n}{a_n} \log n\right)^{\varepsilon}\right) = P\left(X \leq \left(\frac{n}{a_n} \log n\right)^{\varepsilon}\right)$, which implies that

$$\operatorname*{Lim}_{n\to\infty} P\left(T_n \leq a_n^{1/\alpha} \left(\frac{n}{a_n} \log n\right)^{\varepsilon}\right) = 1 \tag{3}$$

Note that

$$P\left(T_n \leq a_n^{1/\alpha} \left(\frac{n}{a_n} \log n\right)^{\varepsilon} \text{ i.o}\right) = P\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\left(T_m \leq a_m^{1/\alpha} \left(\frac{m}{a_m} \log m\right)^{\varepsilon}\right)\right)$$

$$= \operatorname*{Lim}_{n\to\infty} P\left(\bigcup_{m=n}^{\infty} T_m \leq a_m^{1/\alpha} \left(\frac{m}{a_m} \log m\right)^{\varepsilon}\right)$$

$$\geq \operatorname*{Lim}_{n\to\infty} P\left(T_n \leq a_n^{1/\alpha} \left(\frac{n}{a_n} \log n\right)^{\varepsilon}\right)$$

From (3), we get $P\left(T_n \leq a_n^{1/\alpha} \left(\frac{n}{a_n} \log n\right)^{\varepsilon} \text{ i.o}\right) = 1$ and hence the proof of (1) is complete.

Now we will complete the proof of the Lemma by showing that for any $\varepsilon \in (0, 1)$,

$P\left(T_n \leq a_n^{1/\alpha} \left(\frac{n}{a_n} \log n\right)^{-\varepsilon} \text{ i.o}\right) = 0$. We define n_{k+1} as the smallest integer greater than or equal to $n_k + \frac{a_{n_k}}{\log\log a_{n_k}}$, $k = 1, 2, \ldots$; and n_1 as first integer n such that $a_n > 3$. Let $C_{1,n}$, $D_{1,k}$ and $E_{1,k}$ denote the

events

$$C_n = \left\{S_{n+a_n} - S_n \leq a_n^{1/\alpha}\left(\frac{n}{a_n} \log n\right)^{-\varepsilon}\right\},$$

$$D_k = \left\{\inf_{n_k \leq n \leq n_{k+1}} S_{n+a_n} - S_n \leq a_{n_{k+1}}^{1/\alpha}\left(\frac{n_{k+1}}{a_{n_{k+1}}} \log n_{k+1}\right)^{-\varepsilon}\right\} \text{ and}$$

$E_k = \left\{S_{n_k + a_{n_k}} - S_{n_{k+1}} \leq a_{n_{k+1}}^{1/\alpha}\left(\frac{n_{k+1}}{a_{n_{k+1}}} \log n_{k+1}\right)^{-\varepsilon}\right\}$. Notice that $(C_k \text{ i.o})$ $(C_k \text{ i.o}) \subset (D_k \text{ i.o}) \subset (E_k \text{ i.o})$. Hence in order to prove (2), it is enough if we show that

$$P(E_k \text{ i.o}) = 0 \tag{4}$$

We have,

$$P(E_k) = P\left(S_{n_k + a_{n_k}} - S_{n_{k+1}} \leq a_{n_{k+1}}^{1/\alpha}\left(\frac{n_k}{a_{n_k}} \log n_k\right)^{-\varepsilon}\right) =$$

$$P\left(X_1 \leq \frac{a_{n_{k+1}}^{1/\varepsilon}}{\left(n_k + a_{n_k} - n_{k+1}\right)^{1/\alpha}}\left(\frac{n_k}{a_{n_k}} \log n_k\right)^{-\varepsilon}\right).$$

We observe that for $k \geq k_0$

$$\frac{a_{n_{k+1}}^{1/\alpha}}{\left(n_k + a_{n_k} - n_{k+1}\right)^{1/\alpha}}\left(\frac{n_k}{a_{n_k}} \log n_k\right)^{-\varepsilon} \leq 2\left(\frac{a_{n_{k+1}}}{a_{n_k}}\right)^{1/\alpha}\left(\frac{n_{k+1}}{n_k} \log n_{k+1}\right)^{-\varepsilon}.$$

The fact that a_n/n is non – increasing as $n \to \infty$ implies that $\frac{a_{n_{k+1}}}{n_{k+1}} \leq \frac{a_{n_k}}{n_k}$ or $\frac{a_{n_{k+1}}}{a_{n_k}} \leq \frac{n_{k+1}}{n_k}$. Again from the relation

$n_{k+1} = n_k + \frac{a_{n_k}}{\log\log a_{n_k}}$, one can show that $\frac{n_{k+1}}{n_k} \to 1$ as $k \to \infty$. Hence for a given $\varepsilon_1 > 0$ there exists k_1 such that $\frac{a_{n_{k+1}}}{a_{n_k}} \le (1+\varepsilon_1)$, for all $k \ge k_1$.

Consequently for all $k \ge k_1$,

$P(E_k) \le P\left(X_1 \le (1+\varepsilon_1)\left(\frac{n_k}{a_{n_k}} \log n_k\right)^{-\varepsilon}\right)$. From theorem 1 of [6, p.424], one can now get

$P(E_k) \le C \exp\left\{-\left((1+\varepsilon_1)\left(\frac{n_k}{a_{n_k}} \log n_k\right)^{-\varepsilon}\right)^{-\alpha}\right\}$, for some $C > 0$.

Let $(1+\varepsilon_1)^{-\alpha} = (1-\varepsilon_2)$, $\varepsilon_2 > 0$. Then

$$P(E_k) \le C \exp\left\{-(1-\varepsilon_2)\left(\frac{n_k}{a_{n_k}} \log n_k\right)^{\alpha\varepsilon}\right\}.$$

We now claim that, for some $\varepsilon_3 > 0$, $\exp\left\{-(1-\varepsilon_2)\left(\frac{n_k}{a_{n_k}} \log n_k\right)^{\alpha\varepsilon}\right\} = o\left(\frac{a_{n_k}}{n_k} \frac{1}{(\log n_k)^{(1+\varepsilon_3)}}\right)$.

We have

$$\frac{\exp\left\{-(1-\varepsilon_2)\left(\frac{n_k}{a_{n_k}} \log n_k\right)^{\alpha\varepsilon}\right\}}{\frac{a_{n_k}}{n_k} \frac{1}{(\log n_k)^{(1+\varepsilon_3)}}} = \frac{\frac{n_k}{a_{n_k}} (\log n_k)^{(1+\varepsilon_3)}}{\exp\left\{(1-\varepsilon_2)\left(\frac{n_k}{a_{n_k}} \log n_k\right)^{\alpha\varepsilon}\right\}}.$$

The fact that

$\frac{n_k}{a_{n_k}} (\log n_k)^{(1+\varepsilon_3)} \to \infty$ immediately implies that $\frac{\frac{n_k}{a_{n_k}} (\log n_k)^{(1+\varepsilon_3)}}{\exp\left\{(1-\varepsilon_2)\left(\frac{n_k}{a_{n_k}} \log n_k\right)^{\alpha\varepsilon}\right\}} \to 0$ as $n \to \infty$ and the claim is justified. Hence there exists some C_1 (>C) and k_2 such that for all $k \ge k_2$,

$$P(E_k) \le C_1 \left(\frac{a_{n_k}}{n_k}\right)\left(\frac{1}{(\log n_{k+1})^{(1+\varepsilon_3)}}\right) = C_1 \frac{a_{n_k}}{n_k} \frac{1}{(\log n_k)^{(1+\varepsilon_3)}}.$$

Now $n_{k+1} = n_k + \frac{a_{n_k}}{\log\log a_{n_k}}$ gives $a_{n_k} = (n_{k+1} - n_k) \log\log a_{n_k}$.

Using the fact that $\frac{\log\log a_{n_k}}{(\log n_k)^{\frac{\varepsilon_3}{2}}} \to 0$, as $n \to \infty$. We can find a $k_3 \ge k_2$ such that for all $k \ge k_3$,

$$P(E_k) \le C_1 \left(\frac{n_{k+1} - n_k}{n_k}\right)\left(\frac{\log\log a_{n_k}}{(\log n_k)^{(1+\varepsilon_3)}}\right) \le C_1 \left(\frac{n_{k+1} - n_k}{n_k}\right)\left(\frac{1}{(\log n_k)^{\left(1+\frac{\varepsilon_3}{2}\right)}}\right).$$

From the relation $\frac{n_{k+1}}{n_k} \to 1$

as $k \to \infty$ one gets $\int_{k_3}^{\infty} \frac{dt}{t(\log t)^{\left(1+\frac{\varepsilon_3}{2}\right)}} \ge \sum_{k=k_3}^{\infty} \frac{n_{k+1} - n_k}{n_k (\log n_k)^{\left(1+\frac{\varepsilon_3}{2}\right)}}$. Since $\int_{k_3}^{\infty} \frac{dt}{t(\log t)^{\left(1+\frac{\varepsilon_3}{2}\right)}} < \infty$, one gets $\sum_{k=k_3}^{\infty} \frac{n_{k+1} - n_k}{n_k (\log n_k)^{\left(1+\frac{\varepsilon_3}{2}\right)}} < \infty$

or $\sum_{k=k_3}^{\infty} P(E_{1,k}) < \infty$, which in turn establishes (4) by appealing to B.C lemma. Hence the proof of the Lemma is complete.

Lemma 5

$$\operatorname*{Lim\ Sup}_{n \to \infty} \left(\frac{T_n}{a_n^{1/\alpha}}\right)^{\gamma_n} = e^{1/\alpha} \quad \text{a.s}$$

Proof

The proof is on lines of [7] and hence is omitted.

3. LIMIT POINTS OF THE SEQUENCE $\{\xi_n, n \ge 1\}$

We observe by lemmas 3, 4 and 5 that the set of a.s. limit points of the sequences (ξ_n) is included in $[1, e^{1/\alpha}] \times [1, e^{1/\alpha}]$. We will devote this section for the identification of the limit sets of the sequence $\{\xi_n, n \ge 1\}$, when $a_n = [n^p]$, $0 < p < 1$, $a_n = [np]$, $0 < p < 1$ and $a_n = \left[\frac{n}{(\log n)^q}\right]$, $q > 0$. Define $A_1 = \left\{\left[1, e^{\frac{1}{\alpha}}\right] X \left[1, e^{\frac{1}{\alpha}}\right]\right\}$ and $A_2 = \left\{\left(e^{\frac{u}{\alpha}}, e^{\frac{v}{\alpha}}\right) : 0 < u, v < 1,\ u + v \le 1\right\}$.

We will show that a.s. limit set of $\{\xi_n, n \ge 1\}$ coincide with A_1, and A_2 respectively, when $a_n = [n^p]$, $0 < p < 1$, $a_n = [np]$, $0 < p < 1$. When $a_n = \left[\frac{n}{(\log n)^q}\right]$, $q > 0$, we show that the a.s. limit set is again A_2. Hence the limit sets change with the rate of growth of a_n in comparison with n.

Theorem 1

When $a_n = [n^p]$, $0 < p < 1$, the set of all a.s. limit points of the sequence $\{\xi_n, n \ge 1\}$ coincides with $A_1 = \left\{\left[1, e^{\frac{1}{\alpha}}\right] X \left[1, e^{\frac{1}{\alpha}}\right]\right\}$.

Proof

The fact that the limit set of the sequence $\{\xi_n, n \ge 1\}$ is contained in A_1 is immediate from the lemmas 3, 4 and 5.

Hence the proof will be complete once we establish that every element of A_1 is a limit point of $\{\xi_n, n \geq 1\}$. In other words for $\left(e^{\frac{u}{\alpha}}, e^{\frac{v}{\alpha}}\right) \in A_1$ with $0 < u, v < 1$ and $0 < \varepsilon < \min(u, v)$, we have to show that,

$$P\left(\xi_n \in \left(e^{\frac{u-\varepsilon}{\alpha}}, e^{\frac{u+\varepsilon}{\alpha}}\right) X \left(e^{\frac{v-\varepsilon}{\alpha}}, e^{\frac{v+\varepsilon}{\alpha}}\right) \text{i.o}\right) = 1 \tag{5}$$

Note that $a_n = [n^p]$, $0 < p < 1$ implies that $\gamma_n \approx (1\text{-}p)\log n$. We prove that for some $d>0$, $P\left(\xi_n \in \left(e^{\frac{u-\varepsilon}{\alpha}}, e^{\frac{u+\varepsilon}{\alpha}}\right) X \left(e^{\frac{v-\varepsilon}{\alpha}}, e^{\frac{v+\varepsilon}{\alpha}}\right) \text{i.o}\right) \geq d > 0$. This is done by applying lemma1. By Hewitt – Savage zero – one law (5) will be established.

Define $$n_k = \left[k^{\frac{1}{(1-p)v}}\right] \tag{6}$$

and

$$H_k = \left\{ n_k^{1/\alpha}\left(\log n_k\right)^{\frac{u-\varepsilon}{\alpha}} \leq S_{n_k} \leq n_k^{1/\alpha}\left(\log n_k\right)^{\frac{u+\varepsilon}{\alpha}}, \; n_k^{\frac{p}{\alpha}+\frac{v(1-p)}{\alpha}} \leq T_{n_k} \leq n_k^{\frac{p}{\alpha}+\frac{(v+\varepsilon)(1-p)}{\alpha}} \right\}$$

Since X_i's are positive valued strictly stable r.v.s and (S_n) and (T_n) are independent, we have

$$P(H_k) = P\left(\left(\log n_k\right)^{\frac{u-\varepsilon}{\alpha}} \leq X_1 \leq \left(\log n_k\right)^{\frac{u+\varepsilon}{\alpha}}\right) P \left(n_k^{\frac{v(1-p)}{\alpha}} \leq X_2 \leq n_k^{\frac{(v+\varepsilon)(1-p)}{\alpha}} \right)$$

Since X_i's are positive strictly stable r.v.s, we have $P(X \geq x) \sim O(x^{-\alpha})$ (7)

Using (7), we note that there exists constant C_1 (>0) and k_1 such that for all $k \geq k_1$,

$$P\left(H_k\right) \geq \frac{C_1}{k(\log k)^{(u-\varepsilon)}} \tag{8}$$

which implies that $\sum_{k=k_1}^{\infty} P(H_k) = \infty$. Observe that

$$\frac{\sum_{k=1}^{n}\sum_{s=1}^{n} P(H_k \cap H_s)}{\left(\sum_{k=1}^{n} P(H_k)\right)^2} = \frac{2\sum_{k=1}^{n}\sum_{s=k+1}^{n-1} P(H_k \cap H_s)}{\left(\sum_{k=1}^{n} P(H_k)\right)^2} + \frac{1}{\sum_{k=1}^{n} P(H_k)} \tag{9}$$

In order to establish (9) of E.B.C lemma, we proceed as under. For $s > k$, let

$$L_1 = \left\{ n_k^{1/\alpha}\left(\log n_k\right)^{\frac{u-\varepsilon}{\alpha}} \leq S_{n_k} \leq n_k^{1/\alpha}\left(\log n_k\right)^{\frac{u+\varepsilon}{\alpha}} \right\},$$

$$L_2 = \left\{ n_k^{\frac{p}{\alpha}+\frac{v(1-p)}{\alpha}} \leq T_{n_k} \leq n_k^{\frac{p}{\alpha}+\frac{(v+\varepsilon)(1-p)}{\alpha}} \right\}$$

$$L_3 = \left\{ n_s^{1/\alpha}\left(\log n_s\right)^{\frac{u-\varepsilon}{\alpha}} \leq S_{n_s} \leq n_s^{1/\alpha}\left(\log n_s\right)^{\frac{u+\varepsilon}{\alpha}} \right\},$$

$$L_4 = \left\{ \begin{array}{l} n_s^{1/\alpha}\left(\log n_s\right)^{\frac{u-\varepsilon}{\alpha}} - \left(n_k^{1/\alpha}\left(\log n_k\right)^{\frac{u+\varepsilon}{\alpha}} + n_k^{\frac{p}{\alpha}+\frac{(v+\varepsilon)}{\alpha}(1-p)} \right) \leq S_{n_s} - \left(S_{n_k} + T_{n_k}\right) \leq \\ n_s^{1/\alpha}\left(\log n_s\right)^{\frac{u-\varepsilon}{\alpha}} - \left(n_k^{1/\alpha}\left(\log n_k\right)^{\frac{u+\varepsilon}{\alpha}} + n_k^{\frac{p}{\alpha}+\frac{v}{\varepsilon}(1-p)} \right) \end{array} \right\}$$

$$L_5 = \left\{ n_s^{\frac{p}{\alpha}+\frac{v(1-p)}{\alpha}} \leq T_{n_s} \leq n_s^{\frac{p}{\alpha}+\frac{(v+\varepsilon)(1-p)}{\alpha}} \right\}.$$

We have $E_k \cap E_s = \{L_1 \cap L_2 \cap L_3 \cap L_5\} \subset \{L_1 \cap L_2 \cap L_4 \cap L_5\}$ (10)

Since $P(L_3) = P\left(\left(\log n_s\right)^{\frac{u-\varepsilon}{\alpha}} \leq X_1 \leq \left(\log n_s\right)^{\frac{u+\varepsilon}{\alpha}}\right)$. Using (7), one can find constants C_2, k_2 such that for all $k \geq k_2$,

$$P\left(L_3\right) \geq \frac{C_2}{\left(\log n_s\right)^{u-\varepsilon}} \tag{11}$$

Let $s > k(\log k)^{\lambda}$, λ is sufficiently small compared to (1-u), we have

$$P(L_4) \leq P\left(S_{n_s} - \left(S_{n_k} + T_{n_k}\right) \geq n_s^{1/\alpha}\left(\log n_s\right)^{\frac{u-\alpha}{\alpha}} \right)$$

$$= P\left(\frac{S_{n_s} - \left(S_{n_k} + T_{n_k}\right)}{\left(n_s - (n_k + n_k^p)\right)^{1/\alpha}} \geq \frac{n_s^{1/\alpha}\left(\log n_s\right)^{\frac{u-\varepsilon}{\alpha}}}{\left(n_s - (n_k + n_k^p)\right)^{1/\alpha}} \right) \leq P\left(X_1 \geq \frac{n_s^{1/\alpha}\left(\log n_s\right)^{\frac{u-\varepsilon}{\alpha}}}{\left(1 - (n_k + n_k^p)\, n_s^{-1}\right)^{1/\alpha}} \right).$$

Using the fact that $s > k(\log k)^{\lambda}$ one can note that $\frac{n_k + n_k^p}{n_s} \to 0$ as $k \to \infty$, one can find a $C_3 > 0$ and k_3 such that that for all $k \geq k_3$, $P(L_4) \leq \frac{C_3}{(\log n_s)^{(u-\varepsilon)}}$ (12)

From (11) and (12) one can notice that there exists a constant $C_4 > 0$ such that for all

$k \geq k_4 = \max(k_2, k_3)$, $P(L_4) \leq C_4\, P(L_3)$. (13)

From (9) we have for $s > k(\log k)^{\lambda}$ and for $k \geq k_4$,

$$P(H_k \cap H_s) \leq P(L_1 \cap L_2 \cap L_4 \cap L_5) = P(L_1 \cap L_2)P(L_4)P(L_5)$$
$$\leq C_4\; P(H_k)P(L_3)P(L_5)$$

$$\therefore P(H_k \cap H_s) \leq C_4\; P(H_k)P(H_s) \tag{14}$$

Now for $(k+1) \leq s \leq k(\log k)^{\lambda}$, using the inequality $P(H_k \cap H_s) \geq P(H_k \cap L_5)$ and observing that $\left(S_{n_k}\right)$, $\left(T_{n_k}\right)$ and $\left(T_{n_s}\right)$ are independent, one gets $P(H_k \cap H_s) \leq P(H_k)P(L_5)$.

Again using (7) and the fact $s \geq k+1$ one can find a constant $C_5 > 0$ and k_5 such that for all $k \geq k_5$, $P(L_5) \leq \frac{C_5}{k}$.

Hence for all $k \geq k_5$, $P(H_k \cap H_s) \leq \frac{C_5}{k} P(H_k)$ (15)

From (8) note that

$P(H_k) \leq P\left(X_1 \geq (\log n_k)^{\frac{u-\varepsilon}{\alpha}}\right) P\left(X_2 \geq n_k^{\frac{v(1-p)}{\alpha}}\right)$. By applying (7) in (15) one can find constants $C_6 > 0$ and k_6 such that for all $k \geq k_6$, $P(H_k \cap H_s) \leq \frac{C_6}{k^2 (\log k)^{(u-\hat{a})}}$.

Now $\sum_{k=k_6}^{n-1} \sum_{s=k+1}^{k(\log k)^\lambda} P(H_k \cap H_s) \leq C_6 \sum_{k=k_6}^{n-1} \frac{k(\log k)^\lambda}{k^2 (\log k)^{(u-\varepsilon)}} < C_6 \sum_{k=k_6}^{n-1} \frac{1}{k (\log k)^{(u-\varepsilon-\lambda)}}$.

For $n \geq N_1$, we have $\sum_{k=k_6}^{n-1} \sum_{s=k+1}^{k(\log k)^\lambda} P(H_k \cap H_s) \leq C_6 (\log n)^{1-(u-\varepsilon-\lambda)}$. From (8) we have, for $n \geq N_2$, (16)

$\sum_{k=k_6}^{n-1} P(H_k) \geq \sum_{k=k_6}^{n} \frac{C_1}{k(\log k)^{(u-\varepsilon)}} \geq C_7 (\log n)^{1-(u-\varepsilon)}$, for some $C_7 > 0$. (17)

From (14), (16) and (17) one can get $C_8 > 0$,

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{s=1}^{n} P(H_k \cap H_s)}{\left(\sum_{k=1}^{n} P(H_k)\right)^2} \leq C_8. \tag{18}$$

In view of (9), appealing to lemma 1 and Hewitt – Savage zero – one law one gets $P(H_k \text{ i.o}) = 1$. Hence the proof of the theorem is completed.

Theorem 2

When $a_n = [np]$, $0 < p < 1$, the set of all a.s. limit points of the sequence $\{\xi_n, n \geq 1\}$ coincides with A_2.

Proof

Here $\gamma_n \approx \log\log n$. Hence to prove the assertion it is enough to show that for any $\varepsilon > 0$, $u > 0$ and $v > 0$,

$$P\left(\xi_n \in \left(e^{\frac{u-\varepsilon}{\alpha}}, e^{\frac{u+\varepsilon}{\alpha}}; e^{\frac{v-\varepsilon}{\alpha}}, e^{\frac{v+\varepsilon}{\alpha}}\right) \text{i.o}\right) = 0 \tag{19}$$

whenever $u + v > 1$ and

$$P\left(\xi_n \in \left(e^{\frac{u-\varepsilon}{\alpha}}, e^{\frac{u+\varepsilon}{\alpha}}; e^{\frac{v-\varepsilon}{\alpha}}, e^{\frac{v+\varepsilon}{\alpha}}\right) \text{i.o}\right) = 1 \tag{20}$$

whenever $u + v \leq 1$.

Let (u, v) be such that $u + v > 1$. Define $n_k = [e^k]$ and denote the events

$$A_n = \left\{S_n \geq n^{1/\alpha} (\log n)^{\frac{u-\varepsilon}{\alpha}}, S_{n+a_n} - S_n \geq p^{1/\alpha} n^{1/\alpha} (\log n)^{\frac{v-\varepsilon}{\alpha}}\right\},$$

$A_{1,n} = \left\{S_n \geq n^{1/\alpha} (\log n)^{\frac{u-\varepsilon}{\alpha}}\right\}$ and

$$A_{2,n} = \left\{S_{n+a_n} - S_n \geq p^{1/\alpha} n^{1/\alpha} (\log n)^{\frac{v-\varepsilon}{\alpha}}\right\}.$$

Observe that $A_n = \{A_{1,n} \cap A_{2,n}\}$. To show (19), we appeal to Lemma 2. Since (S_n) and $(S_{n+a_n} - S_n)$ are independent, we have

$P(A_n) = P\left(X_1 \geq (\log n)^{\frac{u-\varepsilon}{\alpha}}\right) P\left(X_2 \geq (\log n)^{\frac{v-\varepsilon}{\alpha}}\right)$. Choose $\in$ such that $u+v-2\varepsilon > 1$ and using (7), we get for some C_1 (>0) constant, $P(A_n) \leq \frac{C_1}{(\log n)^{u+v-2\hat{a}}}$. Hence $P(A_n) \to 0$ as $n \to \infty$.

We have

$$(A_n \cap A_{n+1}^c) = A_n \cap (A_{1,n+1}^c \cup A_{2n+1}) = (A_n \cap A_{1,n+1}^c) \cup (A_n \cap A_{2,n+1}^c) \tag{21}$$

Note that $(A_n \cap A_{1,n+1}^c) =$

$$\left\{\begin{array}{l} S_n > n^{1/\alpha} (\log n)^{\frac{u-\varepsilon}{\alpha}}, S_{n+a_n} - S_n > p^{1/\alpha} n^{1/\alpha} (\log n)^{\frac{v-\varepsilon}{\alpha}}, \\ S_{n+1} < (n+1)^{1/\alpha} (\log (n+1))^{\frac{u-\varepsilon}{\alpha}} \end{array}\right\}$$

$$\subseteq \left\{\begin{array}{l} S_n > n^{1/\alpha} (\log n)^{\frac{u-\varepsilon}{\alpha}}, S_{n+a_n} - S_n > p^{1/\alpha} n^{1/\alpha} (\log n)^{\frac{v-\varepsilon}{\alpha}}, \\ S_n < (n+1)^{1/\alpha} (\log (n+1))^{\frac{u-\varepsilon}{\alpha}} \end{array}\right\}$$

$$= \left\{\begin{array}{l} n^{1/\alpha} (\log n)^{\frac{u-\varepsilon}{\alpha}} < S_n < (n+1)^{1/\alpha} (\log (n+1))^{\frac{u-\varepsilon}{\alpha}}, \\ S_{n+a_n} - S_n > p^{1/\alpha} n^{1/\alpha} (\log n)^{\frac{v-\varepsilon}{\alpha}} \end{array}\right\}. \text{ Hence}$$

$$P(A_n \cap A_{1,n+1}^c) \leq P\left((\log n)^{\frac{u-\varepsilon}{\alpha}} < X_1 < \left(\frac{n+1}{n}\right)^{1/\alpha} (\log (n+1))^{\frac{u-\varepsilon}{\alpha}}, X_2 > p^{1/\alpha} (\log n)^{\frac{v-\varepsilon}{\alpha}}\right)$$

$$\leq P(u_n < X_1 < v_n) P\left(X_2 > p^{1/\alpha} (\log n)^{\frac{v-\varepsilon}{\alpha}}\right)$$

where $u_n = (\log n)^{\frac{u-\varepsilon}{\alpha}}$ and $v_n = \left(1 + \frac{1}{n}\right)^{1/\alpha} (\log (n+1))^{\frac{u-\varepsilon}{\alpha}}$. Hence we have,

$P\left(A_n \cap A^c_{1,n+1}\right) \le \int_{u_n}^{v_n} f(x)\,dx \dfrac{C_2}{\left(\log n\right)^{(v-\varepsilon)}}$, where f is the density function of a positive strictly stable r.v, we have density of positive stable law given by $f(x)=\dfrac{C_3}{x^{1+\alpha}}+\dfrac{C_4}{x^{1+2\alpha}}+o\left(\dfrac{1}{x^{1+2\alpha}}\right)$, where $C_3 > 0$ and $C_4 > 0$ are constants. Hence for x large, one can find C > 0 such that $f(x) \le C\left(\dfrac{1}{x^{1+\alpha}}+\dfrac{1}{x^{1+2\alpha}}\right)$. Consequently for n large,

$$P\left(A_n \cap A^c_{1,n+1}\right) \le \frac{C}{\left(\log n\right)^{v-\alpha}} \int_{u_n}^{v_n} \left(\frac{1}{x^{1+\alpha}}+\frac{1}{x^{2+\alpha}}\right)dx$$

$$\le \frac{C}{\left(\log n\right)^{v-\alpha}} \left\{\frac{1}{\alpha}\left(\frac{1}{u_n^{\alpha}}-\frac{1}{v_n^{\alpha}}\right)+\frac{1}{2\alpha}\left(\frac{1}{u_n^{2\alpha}}-\frac{1}{v_n^{2\alpha}}\right)\right\} \quad (22)$$

We have $u_n^{-\alpha}-v_n^{-\alpha}=\left(\log n\right)^{-(u-\varepsilon)}-\left(1+\dfrac{1}{n}\right)^{-1}\left(\log n\left(1+\dfrac{1}{n}\right)\right)^{-(u-\varepsilon)}$

$$=\frac{1}{(\log n)^{u-\varepsilon}}\left(1-\left(1-\frac{1}{n}+\frac{C}{n^2}\right)\frac{(\log n)^{(u-\varepsilon)}}{\left(\log n+\log\left(1+\frac{1}{n}\right)^{(u-\varepsilon)}\right)}\right) \sim \frac{1}{n\left(\log n\right)^{(u-\varepsilon)}} \quad (23)$$

On similar lines one can show that $\dfrac{1}{u_n^{2\alpha}}-\dfrac{1}{v_n^{2\alpha}} \le \dfrac{C_8}{n(\log n)^{(u-\varepsilon)}} \sim \dfrac{C}{n(\log n)^{2u-\varepsilon}}$. (24)

For n large say n ≥ N, from (22) one can show that

$P\left(A_n \cap A^c_{1,n+1}\right) \le \dfrac{C}{n\left(\log n\right)^{(u+v-2\varepsilon)}}$. Since u + v - 2ε > 1, we have

$$\sum_{n=1}^{\infty} P\left(A_n \cap A^c_{1,n+1}\right) \le C_9 \sum_{n=1}^{\infty} \frac{1}{n\left(\log n\right)^{(u+v-2\varepsilon)}} < \infty,$$ for some $C_9 > 0$ (25)

Again following similar lines, one can show that $\sum_{n=1}^{\infty} P\left(A_n \cap A^c_{2,n+1}\right) < \infty$. (26)

Using (25) and (26) in (21), it follows that $\sum_{n=1}^{\infty} P\left(A_n \cap A^c_{n+1}\right) < \infty$ and hence $P\left(A_n \cap A^c_{n+1}\ \text{i.o}\right)=0$, which implies the proof of (21) by Lemma 2.

Defining $n_k=\left[e^{k^{\frac{1}{u+v}}}\right]$ and following the lines of proof similar to those of Theorem 1, the proof of (20) can be obtained and the details are omitted.

Theorem 3

When $a_n=\left[\dfrac{n}{(\log n)^q}\right]$, q > 0, the set of all a.s. limit points of the sequence $\left\{\xi_n, n \ge 1\right\}$ coincides with A_2.

Proof

Here $\gamma_n \approx (1+q)\log\log n$ and from lemmas 4 and 5, we know that

$$\operatorname*{Lim\ Inf\ (Sup)}_{n\to\infty} \left(\frac{T_n}{a_n^{1/\alpha}}\right)^{\frac{1}{(1+q)\log\log n}} = 1\left(e^{1/\alpha}\right) \text{a.s}.$$

Hence $\xi_n=\left\{\left(\dfrac{S_n}{n^{1/\alpha}}\right)^{\theta_n}, \left(\dfrac{T_n}{a_n^{1/\alpha}}\right)^{\frac{\theta_n}{(1+q)}}\right\}$.

Proceeding on the lines of Theorem 2, the a.s. limit set can be shown to be A_2. The details are omitted.

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