On an Asymptotical Moments Problem

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Abstract: In this article we shall give some necessary and sufficient conditions for a discrete law F such that the asymptotic equivalence

$$EX_n^\mu \sim (EX_n)^\mu \ (EX_n \to \infty)$$

takes place for all positive μ. An extension to the class of regularly varying moments is also given.

1. INTRODUCTION

For a sequence of random variables (X_n) define a discrete probability law F by

$$P\{X_n = k\} = p_{nk} \geq 0, \quad k = 0, 1, 2, \cdots, n; \quad \sum_{k=0}^{n} p_{nk} = 1.$$

Define as usual the expectation EX_n and variance $\sigma^2 X_n$ as

$$EX_n := \sum_{k=1}^{n} k p_{nk}; \quad \sigma^2 X_n := EX_n^2 - (EX_n)^2;$$

moments of the m-th order are

$$EX_n^m := \sum k^m p_{nk}, \quad m = 2, 3, \cdots.$$

The question of moments convergence is a difficult one and entirely depends on the characteristics of the law F.

But if

$$EX_n \to \infty \ (n \to \infty)$$

then, due to Jensen's inequality, all other moments are also unbounded and there is the problem of their asymptotic evaluation.

In this paper we shall give some conditions such that the asymptotic equivalence

$$EX_n^\mu \sim (EX_n)^\mu \ (n \to \infty) \tag{1}$$

holds for all real $\mu > 0$, whenever $EX_n \to \infty$ $(n \to \infty)$.

2. RESULTS

It turns out that, under a specific conditions, the validity of the asymptotic relation (1) for some $\mu = m > 1$, implies its validity for all moments of lesser order. This result is characterized in the first proposition.

Proposition 1. *Let $m > 1$ and $EX_n \to \infty$ $(n \to \infty)$*; then the asymptotic equivalence

$$EX_n^\mu \sim (EX_n)^\mu$$

holds for each real μ, $0 < \mu \leq m$, if and only if

$$\limsup_{n\to\infty} \frac{EX_n^m}{(EX_n)^m} \leq 1. \tag{2}$$

Another quite unexpected result is the following

Proposition 2. *If $EX_n \to \infty$ and the probability generating function Es^{X_n},*

$$Es^{X_n} := \sum_{k=0}^{n} p_{nk} s^k,$$

belongs to the class $\mathcal{H}$ of Hurwitz polynomials, then the relation

$$EX_n^\mu \sim (EX_n)^\mu \ (n \to \infty)$$

holds for each $\mu > 0$.

Recall that the class $\mathcal{H}$ consists of all polynomials with non-negative coefficients whose zeros lie entirely in the left complex half-plane (including imaginary axes). This class is of importance in Mechanics and the Theory of Dynamic Stability [1].

Another statement concerning Hurwitz polynomials follows

Proposition 3. Let $H_n(c) := \sum_{k=0}^{n} a_{nk} c^k$, $H_n(c) \in \mathcal{H}$.

For a sequence of random variables (Y_n) define the probability law F_H, with a parameter $c > 0$, as

$$P_H\{Y_n = k\} := \frac{a_{nk} c^k}{H_n(c)}, \quad k = 0, 1, \cdots, n.$$

If, for some $c > 0$, $E_H Y_n = \dfrac{c H_n'(c)}{H_n(c)} \to \infty \quad (n \to \infty)$, then

$$E_H Y_n^\mu \sim (E_H Y_n)^\mu \quad (n \to \infty)$$

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2000 Mathematics Subject Classification 60E05

1876-5270/09 2009 Bentham Open

for each $\mu > 0$.

Further generalization leads to the concept of *regularly varying moments* [2, p. 335].

It is said that the function ℓ is slowly varying in Karamata's sense if it is positive, continuous and satisfies $\ell(\lambda x) \sim \ell(x)$, $(x \to \infty)$ for each $\lambda > 0$ [2, p. 6].

Some examples of slowly varying functions are

1, $(\log x)^a$, $(\log \log x)^b$, $\exp((\log x)^c)$; $a, b \in R$, $0 < c < 1$.

The expression

$$EX_n^\rho \ell(X_n) := \sum_{k=1}^{n} k^\rho \ell(k) p_{nk}$$

is called *regularly varying moment of order* ρ.

For asymptotic behavior of those moments we have

Proposition 4. *If*

(i) $1/EX_n = O(1/n)$; (ii) $\sigma^2(X_n) = o(n^2)$ $(n \to \infty)$,

then the asymptotic relation

$$EX_n^\rho \ell(X_n) \sim (EX_n)^\rho \ell(EX_n) \quad (n \to \infty),$$

holds for each $\rho > 0$ *and every slowly varying function* $\ell(\cdot)$.

3. PROOFS

It is obvious that the condition (2) is necessary for $EX_n^m \sim (EX_n)^m$ to hold. We shall prove that it is also sufficient for the statement of Proposition 1.

Indeed, since $m > 1$, by Jensen's inequality [3, p. 104] it follows that

$$EX_n^m \geq (EX_n)^m.$$

Therefore by (2),

$$1 \leq \liminf_{n\to\infty} \frac{EX_n^m}{(EX_n)^m} \leq \limsup_{n\to\infty} \frac{EX_n^m}{(EX_n)^m} \leq 1.$$

i. e.,

$$EX_n^m \sim (EX_n)^m \quad (n \to \infty). \tag{3}$$

Now we shall prove the case $0 < \mu < m$. For this purpose we need well-known Liapunov moments inequality [3, p. 110].

Lemma 1. *For* $0 < a < b < c$, we have

$$(EX_n^b)^{c-a} \leq (EX_n^a)^{c-b} (EX_n^c)^{b-a}. \tag{4}$$

Applying (4) for $1 < \mu < m$, we get

$$(EX_n^\mu)^{m-1} \leq (EX_n)^{m-\mu} (EX_n^m)^{\mu-1}.$$

By this and Jensen's inequality we obtain

$$1 \leq \frac{EX_n^\mu}{(EX_n)^\mu} \leq \left(\frac{EX_n^m}{(EX_n)^m} \right)^{\frac{\mu-1}{m-1}}. \tag{5}$$

For $0 < \mu < 1$, Jensen's inequality gives

$$EX_n^\mu \leq (EX_n)^\mu.$$

Combining this with (4), ($a = \mu$, $b = 1$, $c = m$) we get

$$1 \geq \frac{EX_n^\mu}{(EX_n)^\mu} \geq \left(\frac{EX_n^m}{(EX_n)^m} \right)^{\frac{-(1-\mu)}{m-1}}. \tag{6}$$

Hence, letting $n \to \infty$, by (3), (5) and (6) we obtain the assertion of Proposition 1.

Proof of Proposition 2

This is a consequence of the assertion from Proposition 3 below, with

$$c := 1;\ a_{nk}/H_n(1) := p_{nk}.$$

Proof of Proposition 3

In the sequel we shall need the following

Lemma 2. *For any* $H \in \mathcal{H}$ *and thus induced law* F_H, *we have*

$$0 \leq \sigma_H^2(Y_n) < 2E_H(Y_n).$$

Proof: From the basic algebra theorems and definition of Hurwitz polynomials, we get the representation

$$H(c) = H_n(c) = a_{nn} \prod_{k\leq n} (c + z_{nk}), \quad a_{nn} > 0, \quad \Re z_{nk} \geq 0.$$

Hence,

$$E_H Y_n = \sum_{k\leq n} k a_{nk} c^k / H(c) = \frac{cH'(c)}{H(c)} = \sum_{k\leq n} \frac{c}{c + z_{nk}} = \sum_{k\leq n} u_{nk} = \sum_{k\leq n} \Re u_{nk},$$

and

$$\sigma_H^2(Y_n) = c\frac{d}{dc}(E_H Y_n) = \sum_{k\leq n} \frac{cz_{nk}}{(c + z_{nk})^2} = \sum_{k\leq n} \upsilon_{nk} = \sum_{k\leq n} \Re \upsilon_{nk},$$

Since $c > 0$, $\Re z_{nk} \geq 0$, we get

$$0 \leq \frac{\Re \upsilon_{nk}}{\Re u_{nk}} = \frac{\Re z_{nk}(c + \Re z_{nk})^2 + (\Im z_{nk})^2 (2c + \Re z_{nk})}{(c + \Re z_{nk})((c + \Re z_{nk})^2 + (\Im z_{nk})^2)} < 2.$$

Therefore,

$$\sigma_H^2(Y_n) = \sum_{k\leq n} \Re \upsilon_{nk} = \sum_{k\leq n} \Re u_{nk} \left(\frac{\Re \upsilon_{nk}}{\Re u_{nk}} \right) < 2\sum_{k\leq n} \Re u_{nk} = 2E_H Y_n,$$

and the proof is done.

Let us consider now a sequence of polynomials $(H_m(c))$, $m = 1, 2, \cdots$ given by the recurrence relation

$$H_m(c) = cH'_{m-1}(c), \qquad H_0(c) = H(c).$$

We have

Lemma 3. *For each* $m \in N$, $H_m(c) \in \mathcal{H}$.

Proof: Suppose that $H_{m-1}(c) \in \mathcal{H}$. By well-known Gauss Theorem, the zeros of $H'_{m-1}(c)$ are not away from the convex polygon enveloping the zeros of $H_{m-1}(c)$. Therefore, all zeros of $cH'_{m-1}(c)$ also belong to the left complex half-plane i. e. $H_m(c) \in \mathcal{H}$. Since $H_0(c) = H(c) \in \mathcal{H}$, the proof follows by induction in m.

Because

$$H_m(c) = \sum_{k\le n} k^m a_{nk} c^k, \quad m \in N,$$

we get

$$(i) E_{H_m} Y_n = \frac{H_{m+1}(c)}{H_m(c)}; (ii) \sigma^2_{H_m}(Y_n) = E_{H_{m+1}} Y_n \cdot E_{H_m} Y_n - (E_{H_m} Y_n)^2.$$

Combining (ii) with Lemmas 2 and 3, we obtain

$$E_{H_{m+1}} Y_n - E_{H_m} Y_n = \sigma^2_{H_m}(Y_n) / E_{H_m} Y_n < 2,$$

i. e. by summing

$$E_{H_m} Y_n < E_H Y_n + 2m, \quad m = 1, 2, \cdots \tag{7}$$

Now, from (i) and (7), it follows that

$$E_H Y_n^m = \frac{H_m(c)}{H(c)} = \prod_{k=0}^{m-1} E_{H_k} Y_n < \prod_{k=0}^{m-1} (E_H Y_n + 2k).$$

Since by assumption $E_H Y_n \to \infty$ $(n \to \infty)$, this gives

$$\limsup_{n\to\infty} \frac{E_H Y_n^m}{(E_H Y_n)^m} \le 1.$$

Hence, by Proposition 1 the assertion from Proposition 3 follows.

Proof of Proposition 4

Note that the conditions (*i*) and (*ii*) imply

$$EX_n \to \infty \ (n \to \infty),$$

and

$$\sigma^2(X_n) = EX_n^2 - (EX_n)^2 = o(n^2) = (n/EX_n)^2 o((EX_n)^2)$$
$$= O(1) o((EX_n)^2) = o((EX_n)^2).$$

Hence

$$EX_n^2 \sim (EX_n)^2 \quad (n \to \infty).$$

Also,

$$o(n^2) = \sigma^2(X_n) = E(X_n - EX_n)^2 \ge (E|X_n - EX_n|)^2;$$

i. e.,

$$E|X_n - EX_n| = o(n) \quad (n \to \infty). \tag{8}$$

We consider firstly the case $\ell(x) = 1$, $\rho \in N$.

Let $q > 2$, $q \in N$. We have

$$0 \le \psi_{n,q} := EX_n^q - EX_n\, EX_n^{q-1}$$
$$= \sum_{k\le n} k^{q-1} p_{nk}(k - EX_n) \le n^{q-1} E|X_n - EX_n|.$$

Applying the condition (*i*) and (8), we get

$$\frac{\psi_{n,q}}{(EX_n)^q} = o(1)(n/EX_n)^q = o(1)O(1) = o(1) \quad (n \to \infty).$$

Finally, by the triangle inequality it follows

$$\left|\frac{EX_n^q}{(EX_n)^q} - 1\right| \le \left|\frac{EX_n^{q-1}}{(EX_n)^{q-1}} - 1\right| + \frac{\psi_{n,q}}{(EX_n)^q},$$

and the proof can be carried out by induction in q.

Since $q > 2$ is arbitrary, by Proposition 1 it follows that the proof is valid for all $\rho > 0$ and $\ell(x) = 1$. For the general case we need the next

Lemma 4. *If the matrix* $\{A_{nk}\}$ *satisfies*

$$(i) \sum_{k\le n} A_{nk} \to 1; \quad (ii) \sum_{k\le n} k^{-v} |A_{nk}| = O(n^{-v}) \quad (n \to \infty)$$

with some $v > 0$, then

$$\sum_{k\le n} A_{nk} \ell(k) \sim \ell(n) \quad (n \to \infty)$$

for each slowly varying $\ell(\cdot)$ (cf [4]).

Putting $A_{nk} := k^\rho p_{nk} / EX_n^\rho$; $v := \rho/2$, we get

$$\sum_{k\le n} A_{nk} = 1;$$
$$\sum_{k\le n} k^{-v} A_{nk} = EX_n^{\rho/2} / EX_n^\rho \sim (EX_n)^{-v} = O(n^{-v}) \quad (n \to \infty)$$

Therefore, by Lemma 4 we have

$$\sum_{k\le n} (k / EX_n)^\rho \ell(k) p_{nk} \sim \ell(n) \quad (n \to \infty).$$

Since $1 \le n/EX_n = O(1)$ $(n \to \infty)$, the Uniform Convergence Theorem [2, p. 6] gives $\ell(n) \sim \ell(EX_n)$, and we finally obtain

$$EX_n^\rho \ell(X_n) := \sum_{k\le n} k^\rho \ell(k) p_{nk} \sim (EX_n)^\rho \ell(EX_n) \quad (n \to \infty).$$

4. EXAMPLES

We shall apply our results on some well-known probability laws. Let us to consider firstly the Gaussian Hypergeometric Law given by

$$P\{X_n = k\} = \binom{M}{k}\binom{N-M}{n-k} \Big/ \binom{N}{k}.$$

Choosing parameters M and N such that

$$N = 2M = 2n + 2A, \quad A \ge 0,$$

we obtain that the probability generating function Es^{X_n},

$$Es^{X_n} = \binom{2n+2A}{n}^{-1} \sum_k \binom{n+A}{k}\binom{n+A}{n-k} s^k.$$

is from the class of *ultraspherical polynomials* [5, pp. 81-86].

Because of orthogonality, all their zeros are real and negative i. e.

$$Es^{X_n} \in \mathcal{H}.$$

Since $EX_n = n/2$, it follows that we can use Proposition 2 to determine asymptotic behavior of all moments of positive order.

Much stronger result can be obtained if we notice that in this case [3]

$$\sigma^2(X_n) = \frac{n(n+2A)}{4(2n+2A-1)} = o(n^2) \quad (n \to \infty).$$

Therefore, applying Proposition 4, we obtain asymptotic behavior of regularly varying moments

$$EX_n^{\rho}\ell(X_n) := \binom{2n+2A}{n}^{-1}\sum_k k^{\rho}\ell(k)\binom{n+A}{k}\binom{n+A}{n-k}$$

$$\sim 2^{-\rho}n^{\rho}\ell(n) \quad (n\to\infty),$$

valid for each slowly varying function $\ell(\cdot)$ and $\rho \in R^{+}$.

Our second example is the classical Binomial Law, defined by

$$P\{X_n = k\} = \binom{n}{k}p^k(1-p)^{n-k}, \quad 0 < p < 1,$$

In this case $EX_n = np$; $\sigma^2(X_n) = np(1-p)$. Hence, by Proposition 4 we get

$$\sum_{k\le n} k^{\rho}\ell(k)\binom{n}{k}p^k(1-p)^{n-k} \sim (np)^{\rho}\ell(np) \sim p^{\rho}n^{\rho}\ell(n)\ (n\to\infty).$$

for each slowly varying $\ell(\cdot)$ and $\rho \in R^{+}$.

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Received: October 30, 2008 Revised: February 23, 2009 Accepted: March 17, 2009